

L^p ESTIMATES FOR BILINEAR AND MULTI-PARAMETER HILBERT TRANSFORMS

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ABSTRACT. C. Muscalu, J. Pipher, T. Tao and C. Thiele proved in [27] that the standard bilinear and bi-parameter Hilbert transform does not satisfy any L^p estimates. They also raised a question asking if a bilinear and bi-parameter multiplier operator defined by

$$T_m(f_1, f_2)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) \hat{f}_1(\xi_1, \eta_1) \hat{f}_2(\xi_2, \eta_2) e^{2\pi i x \cdot ((\xi_1, \eta_1) + (\xi_2, \eta_2))} d\xi d\eta$$

satisfies any L^p estimates, where the symbol m satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma_1)^{|\alpha|}} \cdot \frac{1}{\text{dist}(\eta, \Gamma_2)^{|\beta|}}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, Γ_i ($i = 1, 2$) are subspaces in \mathbb{R}^2 and $\dim \Gamma_1 = 0$, $\dim \Gamma_2 = 1$. P. Silva answered partially this question in [30] and proved that T_m maps $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedly when $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ with $p_1, p_2 > 1$, $\frac{1}{p_1} + \frac{2}{p_2} < 2$ and $\frac{1}{p_2} + \frac{2}{p_1} < 2$. One observes that the admissible range here for these tuples (p_1, p_2, p) is a proper subset contained in the admissible range of BHT.

In this paper, we establish the same L^p estimates as BHT in the full range for the bilinear and multi-parameter Hilbert transforms with arbitrary symbols satisfying appropriate decay assumptions (Theorem 1.3). Moreover, we also establish the same L^p estimates as BHT for certain modified bilinear and bi-parameter Hilbert transforms with $\dim \Gamma_1 = \dim \Gamma_2 = 1$ but with a slightly better decay than that for the bilinear and bi-parameter Hilbert transform (Theorem 1.4).

Keywords: Bilinear and multi-parameter Hilbert transforms; L^p estimates; polydiscs.

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1. INTRODUCTION

The bilinear Hilbert transform is defined by

$$(1.1) \quad BHT(f_1, f_2)(x) := p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t) \frac{dt}{t},$$

or equivalently, it can also be written as the bilinear multiplier operator

$$(1.2) \quad BHT : (f_1, f_2) \mapsto \int_{\xi < \eta} \hat{f}_1(\xi) \hat{f}_2(\eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta,$$

where f_1 and f_2 are Schwartz functions on \mathbb{R} . In [21, 22], M. Lacey and C. Thiele proved the following L^p estimates for bilinear Hilbert transform.

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Theorem 1.1. ([21, 22]) *The bilinear operator BHT maps $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ boundedly for any $1 < p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\frac{2}{3} < r < \infty$.*

There are lots of works related to bilinear operators of BHT type. J. Gilbert and A. Nahmod [10] and F. Bernicot [1] proved that the same L^p estimates as BHT are valid for bilinear operators with more general symbols. Uniform estimates were obtained by C. Thiele [31], L. Grafakos and X. Li [9] and X. Li [23]. A maximal variant of Theorem 1.1 was proved by M. Lacey [20]. In C. Muscalu, C. Thiele and T. Tao [28] and J. Jung [17], the authors investigated various trilinear variants of the bilinear Hilbert transform. For more related results involving estimates for multi-linear singular multiplier operators, we refer to the works, e.g., [3, 4, 5, 8, 11, 12, 16, 19, 25, 26, 32] and the references therein.

In multi-parameter cases, there are also large amounts of literature devoted to studying the estimates of multi-parameter and multi-linear operators (see [2, 6, 7, 14, 18, 24, 25, 27, 29, 30] and the references therein). In the bilinear and bi-parameter cases, let Γ_i ($i = 1, 2$) be subspaces in \mathbb{R}^2 , we consider operators T_m defined by

$$(1.3) \quad T_m(f_1, f_2)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) \hat{f}_1(\xi_1, \eta_1) \hat{f}_2(\xi_2, \eta_2) e^{2\pi i x \cdot ((\xi_1, \eta_1) + (\xi_2, \eta_2))} d\xi d\eta,$$

where the symbol m satisfies¹

$$(1.4) \quad |\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma_1)^{|\alpha|}} \cdot \frac{1}{\text{dist}(\eta, \Gamma_2)^{|\beta|}}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. If $\dim \Gamma_1 = \dim \Gamma_2 = 0$, C. Muscalu, J. Pipher, T. Tao and C. Thiele proved in [27, 29] that Hölder type L^p estimates are available for T_m ; however, if $\dim \Gamma_1 = \dim \Gamma_2 = 1$, let T_m be the double bilinear Hilbert transform on polydisks $BHT \otimes BHT$ defined by

$$(1.5) \quad BHT \otimes BHT(f_1, f_2)(x, y) := p.v. \int_{\mathbb{R}^2} f_1(x - s, y - t) f_2(x + s, y + t) \frac{ds}{s} \frac{dt}{t},$$

they also proved in [27] that the operator $BHT \otimes BHT$ does not satisfy any L^p estimates of Hölder type by constructing a counterexample. In fact, consider bounded functions $f_1(x, y) = f_2(x, y) = e^{ixy}$, one has formally

$$BHT \otimes BHT(f_1, f_2)(x, y) = (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}^2} \frac{e^{2ist}}{st} ds dt = i\pi (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}} \frac{\text{sgn}(s)}{s} ds,$$

then localize functions f_1, f_2 and let $f_1^N(x, y) = f_2^N(x, y) = e^{ixy} \chi_{[-N, N]}(x) \chi_{[-N, N]}(y)$, one can verify the pointwise estimate

$$(1.6) \quad |BHT \otimes BHT(f_1^N, f_2^N)(x, y)| \geq \left| \int_{-\frac{N}{10}}^{\frac{N}{10}} \int_{-\frac{N}{10}}^{\frac{N}{10}} \frac{e^{2ist}}{st} ds dt \right| + O(1) \geq C \log N + O(1)$$

for every $x, y \in [-\frac{N}{100}, \frac{N}{100}]$ and sufficiently large $N \in \mathbb{Z}^+$, which indicates that no Hölder type L^p estimates are available for the bilinear operator $BHT \otimes BHT$. When $\dim \Gamma_1 = 0$

¹Throughout this paper, $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. If necessary, we use explicitly $A \lesssim_{\star, \dots, \star} B$ to indicate that there exists a positive constant $C_{\star, \dots, \star}$ depending only on the quantities appearing in the subscript continuously such that $A \leq C_{\star, \dots, \star} B$.

and $\dim \Gamma_2 = 1$, C. Muscalu, J. Pipher, T. Tao and C. Thiele raised the following problem in Question 8.2 in [27].

Question 1.2. ([27]) *Let $\dim \Gamma_1 = 0$ and $\dim \Gamma_2 = 1$ with Γ_2 non-degenerate in the sense of [26]. If m is a multiplier satisfying (1.4), does the corresponding operator T_m defined by (1.3) satisfy any L^p estimates?*

In [30], P. Silva answered this question partially and proved that T_m defined by (1.3), (1.4) with $\dim \Gamma_1 = 0$ and $\dim \Gamma_2 = 1$ maps $L^p \times L^q \rightarrow L^r$ boundedly when $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $p, q > 1$, $\frac{1}{p} + \frac{2}{q} < 2$ and $\frac{1}{q} + \frac{2}{p} < 2$. One should observe that the admissible range for these tuples (p, q, r) is a proper subset of the region $p, q > 1$ and $\frac{3}{4} < r < \infty$, which is also properly contained in the admissible range of BHT (see Theorem 1.1).

Naturally, we may wonder whether the bi-parameter bilinear operator T_m given by (1.3), (1.4) (with appropriate decay assumptions on the symbol m and singularity sets Γ_1, Γ_2 satisfying $\dim \Gamma_1 = 0$ or 1 , $\dim \Gamma_2 = 1$) satisfies the same L^p estimates as BHT.

To study this problem, we must find the implicit decay assumptions on symbol m to preclude the existence of those kinds of counterexamples constructed in the above (1.6) for $BHT \otimes BHT$. To this end, let us consider first the bilinear operator $T_m \otimes BHT$ of tensor product type, which is defined by

$$(1.7) \quad T_m \otimes BHT(f_1, f_2)(x, y) := p.v. \int_{\mathbb{R}^2} f_1(x-s, y-t) f_2(x+s, y+t) \frac{K(s)}{t} ds dt,$$

where the symbol $m(\xi_1^1, \xi_2^1) = m(\zeta) := \hat{K}(\zeta)$ with $\zeta := \xi_1^1 - \xi_2^1$ has one dimensional non-degenerate singularity set Γ_1 . Let $f_1(x, y) = f_2(x, y) = e^{ixy}$, one can easily derive that

$$(1.8) \quad T_m \otimes BHT(f_1, f_2)(x, y) = (f_1 \cdot f_2)(x, y) \int_{\mathbb{R}^2} K(s) \frac{e^{2ist}}{t} ds dt.$$

From (1.8) and the above counterexample constructed in (1.6) for operator $BHT \otimes BHT$, we observe that one sufficient condition for precluding the existence of these kinds of counterexamples is $K \in L^1$, or equivalently, $m = \hat{K} \in \mathcal{F}(L^1)$. From the Riemann-Lebesgue theorem, we know that a necessary condition for $m \in \mathcal{F}(L^1)$ is $m(\zeta) \rightarrow 0$ as $|\zeta| \rightarrow \infty$. Moreover, if $K \in L^1(\mathbb{R})$ is odd, one can even derive that $|\int_{\mathbb{R}} \frac{m(\zeta)}{\zeta} d\zeta| \lesssim \|K\|_{L^1}$ (this indicates that there are many uniformly continuous functions with logarithmic decay rate do not belong to $\mathcal{F}(L^1)$). Therefore, in order to guarantee that the same L^p estimates as the bilinear Hilbert transform are available for bilinear operators $T_m \otimes BHT$ and $BHT \otimes BHT$, we need some appropriate decay assumptions on the symbol.

The purpose of this paper is to prove the same L^p estimates as BHT for modified bilinear operators $T_m^\varepsilon \otimes BHT$ and $BHT^\varepsilon \otimes BHT$ with arbitrary non-smooth symbols which decay faster than the logarithmic rate.

For $d \geq 2$, any two generic vectors $\xi_1 = (\xi_1^i)_{i=1}^d$, $\xi_2 = (\xi_2^i)_{i=1}^d$ in \mathbb{R}^d generates naturally the following collection of d vectors in \mathbb{R}^2 :

$$(1.9) \quad \bar{\xi}_1 = (\xi_1^1, \xi_2^1), \quad \bar{\xi}_2 = (\xi_1^2, \xi_2^2), \quad \dots, \quad \bar{\xi}_d = (\xi_1^d, \xi_2^d).$$

For arbitrary small $\varepsilon > 0$, let $m^\varepsilon = m^\varepsilon(\xi) = m^\varepsilon(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{2d})$ that is smooth away from the subspaces $\Gamma_1 \cup \cdots \cup \Gamma_{d-1} \cup \Gamma_d$ and satisfying

$$(1.10) \quad \text{dist}(\bar{\xi}_d, \Gamma_d)^{|\alpha_d|} \cdot \int_{\mathbb{R}^{2(d-1)}} \frac{|\partial_{\bar{\xi}_1}^{\alpha_1} \cdots \partial_{\bar{\xi}_d}^{\alpha_d} m^\varepsilon(\bar{\xi})|}{\prod_{i=1}^{d-1} \text{dist}(\bar{\xi}_i, \Gamma_i)^{2-|\alpha_i|}} d\bar{\xi}_1 \cdots d\bar{\xi}_{d-1} \leq B(\varepsilon) < +\infty$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_d$, where the constants $B(\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, $\dim \Gamma_i = 0$ for $i = 1, \dots, d-1$ and $\Gamma_d := \{(\xi_1^d, \xi_2^d) \in \mathbb{R}^2 : \xi_1^d = \xi_2^d\}$. Denote by $T_{m^\varepsilon}^{(d)}$ the bilinear multiplier operator defined by

$$(1.11) \quad T_{m^\varepsilon}^{(d)}(f_1, f_2)(x) := \int_{\mathbb{R}^{2d}} m^\varepsilon(\xi) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi.$$

Our result for bilinear operators $T_{m^\varepsilon}^{(d)}$ satisfying (1.10) and (1.11) is the following Theorem 1.3.

Theorem 1.3. *For any $d \geq 2$ and $\varepsilon > 0$, the bilinear, d -parameter multiplier operator $T_{m^\varepsilon}^{(d)}$ maps $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ boundedly for any $1 < p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. The implicit constants in the bounds depend only on $p_1, p_2, p, \varepsilon, d$ and tend to infinity as $\varepsilon \rightarrow 0$.*

As shown in [27], the bilinear and bi-parameter Hilbert transform does not satisfy any L^p estimates. This is the case when the singularity sets Γ_1 and Γ_2 satisfy $\dim \Gamma_1 = \dim \Gamma_2 = 1$. Thus, it is natural to ask if the L^p estimates will break down for any bilinear and bi-parameter Fourier multiplier operator with $\dim \Gamma_1 = \dim \Gamma_2 = 1$. In other words, will a non-smooth symbol with the same dimensional singularity sets but with a slightly better decay than that for the bilinear and bi-parameter Hilbert transform assure the L^p estimates? Our next theorem will address this issue.

For $d = 2$ and arbitrary small $\varepsilon > 0$, let $\tilde{m}^\varepsilon = \tilde{m}^\varepsilon(\xi) = \tilde{m}^\varepsilon(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^4)$ that is smooth away from the subspaces $\Gamma_1 \cup \Gamma_2$ and satisfying

$$(1.12) \quad |\partial_{\bar{\xi}_1}^{\alpha_1} \partial_{\bar{\xi}_2}^{\alpha_2} \tilde{m}^\varepsilon(\bar{\xi})| \lesssim \prod_{i=1}^2 \frac{1}{\text{dist}(\bar{\xi}_i, \Gamma_i)^{|\alpha_i|}} \cdot \langle \log_2 \text{dist}(\bar{\xi}_1, \Gamma_1) \rangle^{-(1+\varepsilon)}$$

for sufficiently many multi-indices α_1, α_2 , where $\langle x \rangle := \sqrt{1+x^2}$ and $\Gamma_i := \{(\xi_1^i, \xi_2^i) \in \mathbb{R}^2 : \xi_1^i = \xi_2^i\}$ for $i = 1, 2$. Denote by $T_{\tilde{m}^\varepsilon}^{(2)}$ the bilinear multiplier operator defined by

$$(1.13) \quad T_{\tilde{m}^\varepsilon}^{(2)}(f_1, f_2)(x) := \int_{\mathbb{R}^4} \tilde{m}^\varepsilon(\xi) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi.$$

Our result for bilinear operators $T_{\tilde{m}^\varepsilon}^{(2)}$ satisfying (1.12) and (1.13) is the following Theorem 1.4.

Theorem 1.4. *For $d = 2$ and any $\varepsilon > 0$, the bilinear, bi-parameter multiplier operator $T_{\tilde{m}^\varepsilon}^{(2)}$ maps $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ boundedly for any $1 < p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. The implicit constants in the bounds depend only on p_1, p_2, p, ε and tend to infinity as $\varepsilon \rightarrow 0$. In addition, let the bilinear, bi-parameter operator $BHT^\varepsilon \otimes BHT^\varepsilon$*

be defined by

$$BHT^\varepsilon \otimes BHT(f_1, f_2)(x_1, x_2) = p.v. \int_{\mathbb{R}^2} f_1(x-s)f_2(x+s)\Psi^\varepsilon(s_1)\frac{ds_1}{s_1}\frac{ds_2}{s_2}$$

with the function Ψ^ε satisfying

$$(1.14) \quad |\partial_{\xi_1}^{\alpha_1} \widehat{\Psi^\varepsilon}(\xi_1^1 - \xi_2^1)| \lesssim |\xi_1^1 - \xi_2^1|^{-|\alpha_1|} \cdot \langle \log_2 |\xi_1^1 - \xi_2^1| \rangle^{-(1+\varepsilon)}$$

for sufficiently many multi-indices α_1 , then it satisfies the same L^p estimates as $T_{\tilde{m}^\varepsilon}^{(2)}$.

Remark 1.5. For simplicity, we will only consider the bi-parameter case $d = 2$ and $\Gamma_i = \{(0, 0)\}$ ($i = 1, \dots, d-1$) in the proof of Theorem 1.3. It will be clear from the proof (see Section 4) that we can extend the argument to the general d -parameter and $\dim \Gamma_i = 0$ ($i = 1, \dots, d-1$) cases straightforwardly. In the proof of Theorem 1.4, we will only prove the L^p estimates for bilinear and bi-parameter operators $T_{\tilde{m}^\varepsilon}^{(2)}$, since one can observe from the discretization procedure in Section 2 that the bilinear and bi-parameter operator $BHT^\varepsilon \otimes BHT$ can be reduced to the same bilinear model operators $\tilde{\Pi}_{\mathbb{P}}^\varepsilon$ as $T_{\tilde{m}^\varepsilon}^{(2)}$.

It's well known that a standard approach to prove L^p estimates for one-parameter n -linear operators with singular symbols (e.g., Coifman-Meyer multiplier, BHT and one-parameter paraproducts) is the generic estimates of the corresponding $(n+1)$ -linear forms consisting of estimates for different *sizes* and *energies* (see [17, 25, 26, 28]), which relied on the one dimensional BMO theory, or more precisely, the John-Nirenberg type inequalities to get good control over the relevant *sizes*. Unfortunately, there is no routine generalization of such approach to multi-parameter settings, for instance, we don't have analogues of the John-Nirenberg inequalities for *dyadic rectangular BMO* spaces in two-parameter case (see [25]). To overcome these difficulties, in [27] C. Muscalu, J. Pipher, T. Tao and C. Thiele developed a completely new approach to prove L^p estimates for bi-parameter paraproducts, their essential ideas is to apply the *stopping-time decompositions* based on hybrid square and maximal operators MM , MS , SM and SS , the one dimensional BMO theory and Journé's lemma, and hence could not be extended to solve the general d -parameter ($d \geq 3$) cases. As to the general d -parameter ($d \geq 3$) cases, by proving a generic decomposition (see Lemma 4.1) in [29], the authors simplified the arguments introduced by them in [27] and this simplification works equally well in all d -parameter settings. Recently, a pseudo-differential variant of the theorems in [27, 29] has been established by the current authors in [6]. Moreover, in the work [3] by J. Chen and the second author, they offer a different proof than those in [27, 29] to establish a Hörmander type theorem of L^p estimates (and weighted estimates as well) for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness in multi-parameter Sobolev spaces.

However, in this paper, in order to prove our main Theorems 1.3 and 1.4 in bi-parameter settings, we have at least two different difficulties from [27]. First, observe that if one restricts the sum of *tri-tiles* $P'' \in \mathbb{P}''$ in the definitions of discrete model operators (see Section 2) to a *tree* then one essentially gets a discrete paraproduct on x_2 variable, which can be estimated by the MM , MS , SM and SS functions, but due to the *extra degree of freedom* in frequency in x_2 direction, there are infinitely many such paraproducts in the summation, so it's difficult for us to carry out the *stopping-time decompositions* by using the hybrid square and maximal operators. Second, in the proof of Theorem 1.4, note that

there are infinitely many *tri-tiles* $P' \in \mathbb{P}'$ with the property that $I_{P'} = I_0$ for a certain fixed dyadic interval I_0 of the same length as $I_{P'}$, so we can't get estimate $\sum_{P'} |I_{P'}| \lesssim |\tilde{I}|$ for all dyadic intervals $I_{P'} \subseteq \tilde{I}$ with comparable lengths, and hence we can't apply the Journé's lemma either. By making use of the L^2 sizes and L^2 energies estimates of the tri-linear forms, the *almost orthogonality* of *wave packets* associated with different *tiles* of distinct *trees* and the decay assumptions on the symbols, we are able to overcome these difficulties in the proof of Theorem 1.3 and 1.4 in bi-parameter settings.

Nevertheless, in the proof of Theorem 1.4 in general d -parameter settings ($d \geq 3$), one easily observe that the generic decomposition will destroy the *perfect orthogonality* of *wave packets* associated with distinct *tiles* which have disjoint frequency intervals in both x_1 and x_2 directions, thus we can't apply the generic decomposition to extend the results of Theorem 1.4 to higher parameters $d \geq 3$. For the proof of Theorem 1.3, we are able to apply the generic decomposition lemma (Lemma 4.1) to the $d - 1$ variables x_1, \dots, x_{d-1} . Although one can't obtain that $\text{supp } \Phi_{P'}^{3,\ell} \otimes \Phi_{P''}^3$ is entirely contained in the exceptional set U as in [29], but one can observe that the support set is contained in U in all the x_1, \dots, x_{d-1} variables except the last x_d . Therefore, we only need to consider the distance from the support set to the set E'_3 in x_d direction and obtain enough decay factors for summation, the extension of the proof to the general d -parameter ($d \geq 3$) cases is straightforward.

The rest of this paper is organized as follows. In Section 2 we reduce the proof of Theorem 1.3 and Theorem 1.4 to proving restricted weak type estimates of discrete bilinear model operators $\Pi_{\mathbb{P}}^\varepsilon$ and $\tilde{\Pi}_{\mathbb{P}}^\varepsilon$ (Proposition 2.17). Section 3 is devoted to giving a review of the definitions and useful properties about trees, L^2 sizes and L^2 energies introduced in [28]. In Section 4 and 5 we carry out the proof of Proposition 2.17, which completes the proof of our main theorems, Theorem 1.3 and Theorem 1.4, respectively.

2. REDUCTION TO RESTRICTED WEAK TYPE ESTIMATES OF DISCRETE BILINEAR MODEL OPERATORS $\Pi_{\mathbb{P}}^\varepsilon$ AND $\tilde{\Pi}_{\mathbb{P}}^\varepsilon$

2.1. Discretization. As we can see from the study of multi-parameter and multi-linear Coifman-Meyer multiplier operators (see e.g. [26, 27, 28, 29]), a standard approach to obtain L^p estimates of bilinear operators $T_{m^\varepsilon}^{(d)}$ and $T_{\tilde{m}^\varepsilon}^{(2)}$ is to reduce them into discrete sums of inner products with wave packets (see [32]).

2.1.1. Discretization for bilinear, bi-parameter operators $T_{m^\varepsilon}^{(2)}$ with $\Gamma_1 = \{(0, 0)\}$. We will proceed the discretization procedure as follows. First, we need to decompose the symbol $m^\varepsilon(\xi)$ in a natural way. To this end, for the first spatial variable x_1 , we decompose the region $\{\bar{\xi}_1 = (\xi_1^1, \xi_1^2) \in \mathbb{R}^2 \setminus \{(0, 0)\}\}$ by using *Whitney squares* with respect to the singularity point $\{\xi_1^1 = \xi_1^2 = 0\}$; while for the last spatial variable x_2 , we decompose the region $\{\bar{\xi}_2 = (\xi_2^1, \xi_2^2) \in \mathbb{R}^2 : \xi_2^1 \neq \xi_2^2\}$ by using *Whitney squares* with respect to the singularity line $\Gamma_2 = \{\xi_2^1 = \xi_2^2\}$. In order to describe our discretization procedure clearly, let us first recall some standard notation and definitions in [28].

An interval I on the real line \mathbb{R} is called dyadic if it is of the form $I = 2^{-k}[n, n + 1]$ for some $k, n \in \mathbb{Z}$. An interval is said to be a *shifted dyadic interval* if it is of the form $2^{-k}[j + \alpha, j + 1 + \alpha]$ for any $k, j \in \mathbb{Z}$ and $\alpha \in \{0, \frac{1}{3}, -\frac{1}{3}\}$. A *shifted dyadic cube* is a set of the

form $Q = Q_1 \times Q_2 \times Q_3$, where each Q_j is a shifted dyadic interval and they all have the same length. A *shifted dyadic quasi-cube* is a set $Q = Q_1 \times Q_2 \times Q_3$, where Q_j ($j = 1, 2, 3$) are shifted dyadic intervals satisfying less restrictive condition $|Q_1| \simeq |Q_2| \simeq |Q_3|$. One easily observe that for every cube $Q \subseteq \mathbb{R}^3$, there exists a shifted dyadic cube \tilde{Q} such that $Q \subset \frac{7}{10}\tilde{Q}$ (the cube having the same center as \tilde{Q} but with side length $\frac{7}{10}$ that of \tilde{Q}) and $\text{diam}(Q) \simeq \text{diam}(\tilde{Q})$.

The same terminology will also be used in the plane \mathbb{R}^2 . The only difference is that the previous cubes now become squares.

For any cube and square Q , we will denote the side length of Q by $\ell(Q)$ for short and denote the reflection of Q with respect to the origin by $-Q$ hereafter.

Definition 2.1. ([25, 29]) For $J \subseteq \mathbb{R}$ an arbitrary interval, we say that a smooth function Φ_J is a bump adapted to J , if and only if the following inequalities hold:

$$(2.1) \quad |\Phi_J^{(l)}(x)| \lesssim_{l,\alpha} \frac{1}{|J|^l} \cdot \frac{1}{(1 + \frac{\text{dist}(x,J)}{|J|})^\alpha}$$

for every integer $\alpha \in \mathbb{N}$ and for sufficiently many derivatives $l \in \mathbb{N}$. If Φ_J is a bump adapted to J , we say that $|J|^{-\frac{1}{2}}\Phi_J$ is an L^2 -normalized bump adapted to J .

Now let $\varphi \in \mathcal{S}(\mathbb{R})$ be an even Schwartz function such that $\text{supp } \hat{\varphi} \subseteq [-\frac{3}{16}, \frac{3}{16}]$ and $\hat{\varphi}(\xi) = 1$ on $[-\frac{1}{6}, \frac{1}{6}]$, and define $\psi \in \mathcal{S}(\mathbb{R})$ to be the Schwartz function whose Fourier transform satisfies $\hat{\psi}(\xi) := \hat{\varphi}(\frac{\xi}{4}) - \hat{\varphi}(\frac{\xi}{2})$ and $\text{supp } \hat{\psi} \subseteq [-\frac{3}{4}, -\frac{1}{3}] \cup [\frac{1}{3}, \frac{3}{4}]$, such that $0 \leq \hat{\varphi}(\xi), \hat{\psi}(\xi) \leq 1$. Then, for every integer $k \in \mathbb{Z}$, we define $\widehat{\varphi}_k, \widehat{\psi}_k \in \mathcal{S}(\mathbb{R})$ by

$$(2.2) \quad \widehat{\varphi}_k(\xi) := \hat{\varphi}(\frac{\xi}{2^k}), \quad \widehat{\psi}_k(\xi) := \hat{\psi}(\frac{\xi}{2^k}) = \widehat{\varphi_{k+2}}(\xi) - \widehat{\varphi_{k+1}}(\xi)$$

and observe that

$$\text{supp } \widehat{\varphi}_k \subseteq [-\frac{3}{16} \cdot 2^k, \frac{3}{16} \cdot 2^k], \quad \text{supp } \widehat{\psi}_k \subseteq [-\frac{3}{4} \cdot 2^k, -\frac{1}{3} \cdot 2^k] \cup [\frac{1}{3} \cdot 2^k, \frac{3}{4} \cdot 2^k],$$

and $\text{supp } \widehat{\psi}_k \cap \text{supp } \widehat{\psi}_{k'} = \emptyset$ for any integers $k, k' \in \mathbb{Z}$ such that $|k-k'| \geq 2$, $\text{supp } \hat{\varphi} \cap \text{supp } \widehat{\psi}_k = \emptyset$ for any integer $k \geq 0$. One easily obtain the homogeneous Littlewood-Paley dyadic decomposition

$$(2.3) \quad 1 = \sum_{k \in \mathbb{Z}} \widehat{\psi}_k(\xi), \quad \forall \xi \in \mathbb{R} \setminus \{0\}$$

and inhomogeneous Littlewood-Paley dyadic decomposition

$$(2.4) \quad 1 = \hat{\varphi}(\xi) + \sum_{k \geq -1} \widehat{\psi}_k(\xi), \quad \forall \xi \in \mathbb{R},$$

as a consequence, we get decomposition for the product $1(\xi_1^1, \xi_2^1) = 1(\xi_1^1) \cdot 1(\xi_2^1)$ as follows:

$$(2.5) \quad 1(\xi_1^1, \xi_2^1) = \sum_{k' \in \mathbb{Z}} \widehat{\varphi}_{k'}(\xi_1^1) \widehat{\psi}_{k'}(\xi_2^1) + \sum_{k' \in \mathbb{Z}} \widehat{\psi}_{k'}(\xi_1^1) \widehat{\psi}_{k'}(\xi_2^1) + \sum_{k' \in \mathbb{Z}} \widehat{\psi}_{k'}(\xi_1^1) \widehat{\varphi}_{k'}(\xi_2^1)$$

for every $(\xi_1^1, \xi_2^1) \neq (0, 0)$, where

$$\widehat{\psi_{k'}} := \sum_{|k-k'| \leq 1, k \in \mathbb{Z}} \widehat{\psi_k}, \quad \forall k' \in \mathbb{Z}.$$

By writing the characteristic function of the plane (ξ_1^1, ξ_2^1) into finite sums of smoothed versions of characteristic functions of cones as in (2.5), we can decompose the operator $T_{m^\varepsilon}^{(2)}$ into a finite sum of several parts in x_1 direction. Since all the operators obtained in this decomposition can be treated in the same way, we will discuss in detail only one of them. More precisely, let

$$(2.6) \quad \widetilde{\mathcal{Q}} := \{\widetilde{Q}' = \widetilde{Q}'_1 \times \widetilde{Q}'_2 \subseteq \mathbb{R}^2 : \widetilde{Q}'_1 := 2^{k'}[-\frac{1}{2}, \frac{1}{2}], \widetilde{Q}'_2 := 2^{k'}[\frac{1}{24}, \frac{25}{24}], \forall k' \in \mathbb{Z}\},$$

for each square $\widetilde{Q}' \in \widetilde{\mathcal{Q}}$, we define bump functions $\phi_{\widetilde{Q}'_i, i}$ ($i = 1, 2$) adapted to intervals \widetilde{Q}'_i and satisfying $\text{supp } \phi_{\widetilde{Q}'_i, i} \subseteq \frac{9}{10}\widetilde{Q}'_i$ by

$$(2.7) \quad \phi_{\widetilde{Q}'_1, 1}(\xi) := \widehat{\varphi}(\frac{\xi}{\ell(\widetilde{Q}')}) = \widehat{\varphi_{k'}}(\xi)$$

and

$$(2.8) \quad \phi_{\widetilde{Q}'_2, 2}(\xi) := \widehat{\psi}(\frac{\xi}{\ell(\widetilde{Q}')}) \cdot \chi_{\{\xi > 0\}} = \widehat{\psi_{k'}}(\xi) \cdot \chi_{\{\xi > 0\}},$$

respectively, and finally define smooth bump functions $\phi_{\widetilde{Q}'}$ adapted to \widetilde{Q}' and satisfying $\text{supp } \phi_{\widetilde{Q}'} \subseteq \frac{9}{10}\widetilde{Q}'$ by

$$(2.9) \quad \phi_{\widetilde{Q}'}(\xi_1^1, \xi_2^1) := \phi_{\widetilde{Q}'_1, 1}(\xi_1^1) \cdot \phi_{\widetilde{Q}'_2, 2}(\xi_2^1).$$

Without loss of generality, we will only consider the smoothed characteristic function of the cone $\{(\xi_1^1, \xi_2^1) \in \mathbb{R}^2 : |\xi_1^1| \lesssim |\xi_2^1|, \xi_2^1 > 0\}$ in the decomposition (2.5) from now on, which is defined by

$$(2.10) \quad \sum_{\widetilde{Q}' \in \widetilde{\mathcal{Q}}} \phi_{\widetilde{Q}'}(\xi_1^1, \xi_2^1).$$

As to the x_2 direction, we consider the collection \mathcal{Q}'' of all shifted dyadic squares $Q'' = Q''_1 \times Q''_2$ satisfying

$$(2.11) \quad Q'' \subseteq \{(\xi_1^2, \xi_2^2) \in \mathbb{R}^2 : \xi_1^2 \neq \xi_2^2\}, \quad \text{dist}(Q'', \Gamma_2) \simeq 10^4 \text{diam}(Q'').$$

We can split the collection \mathcal{Q}'' into two disjoint sub-collections, that is, define

$$(2.12) \quad \mathcal{Q}''_{\text{I}} := \{Q'' \in \mathcal{Q}'' : Q'' \subseteq \{\xi_1^2 < \xi_2^2\}\}, \quad \mathcal{Q}''_{\text{II}} := \{Q'' \in \mathcal{Q}'' : Q'' \subseteq \{\xi_1^2 > \xi_2^2\}\}.$$

Since the set of squares $\{\frac{7}{10}Q'' : Q'' \in \mathcal{Q}''\}$ also forms a finitely overlapping cover of the region $\{\xi_1^2 \neq \xi_2^2\}$, we can apply a standard partition of unity and write the symbol $\chi_{\{\xi_1^2 \neq \xi_2^2\}}$ as

$$(2.13) \quad \chi_{\{\xi_1^2 \neq \xi_2^2\}} = \sum_{Q'' \in \mathcal{Q}''} \phi_{Q''}(\xi_1^2, \xi_2^2) = \{ \sum_{Q'' \in \mathcal{Q}''_{\text{I}}} + \sum_{Q'' \in \mathcal{Q}''_{\text{II}}} \} \phi_{Q''}(\xi_1^2, \xi_2^2) = \chi_{\{\xi_1^2 < \xi_2^2\}} + \chi_{\{\xi_1^2 > \xi_2^2\}},$$

where each $\phi_{Q''}$ is a smooth bump function adapted to Q'' and supported in $\frac{8}{10}Q''$.

One can easily observe that we only need to discuss in detail one term in the decomposition (2.13), since the other term can be treated in the same way. Without loss of generality, we will only consider the first term in the decomposition (2.13), that is, the characteristic function $\chi_{\{\xi_1^2 < \xi_2^2\}}$ of the upper half plane with respect to singularity line Γ_2 , which can be written as

$$(2.14) \quad \chi_{\{\xi_1^2 < \xi_2^2\}} = \sum_{Q'' \in \mathbb{Q}_{\mathbb{I}}''} \phi_{Q''}(\xi_1^2, \xi_2^2).$$

In a word, we only need to consider the bilinear operator $T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}$ given by

$$(2.15) \quad T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2)(x) := \sum_{\widetilde{Q}' \in \widetilde{\mathbb{Q}}', Q'' \in \mathbb{Q}_{\mathbb{I}}''} \int_{\mathbb{R}^4} m^\varepsilon(\xi) \phi_{\widetilde{Q}'}(\bar{\xi}_1) \phi_{Q''}(\bar{\xi}_2) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi$$

from now on, and the proof of Theorem 1.3 can be reduced to proving the following L^p estimates for $T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}$:

$$(2.16) \quad \|T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2)\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon, p, p_1, p_2} \|f_1\|_{L^{p_1}(\mathbb{R}^2)} \cdot \|f_2\|_{L^{p_2}(\mathbb{R}^2)},$$

as long as $1 < p_1, p_2 \leq \infty$ and $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$.

On one hand, since $\xi_1^1 \in \text{supp } \phi_{\widetilde{Q}'_{1,1}} \subseteq \ell(\widetilde{Q}')[-\frac{3}{16}, \frac{3}{16}]$ and $\xi_2^1 \in \text{supp } \phi_{\widetilde{Q}'_{2,2}} \subseteq \ell(\widetilde{Q}')[\frac{1}{3}, \frac{3}{4}]$, it follows that $-\xi_1^1 - \xi_2^1 \in \ell(\widetilde{Q}')[-\frac{15}{16}, -\frac{7}{48}]$, and as a consequence, there exists a interval $\widetilde{Q}'_3 := \ell(\widetilde{Q}')[-\frac{25}{24}, -\frac{1}{24}]$ and a bump function $\phi_{\widetilde{Q}'_3,3}$ adapted to \widetilde{Q}'_3 such that $\text{supp } \phi_{\widetilde{Q}'_3,3} \subseteq \ell(\widetilde{Q}')[-\frac{23}{24}, -\frac{1}{8}] \subseteq \frac{9}{10}\widetilde{Q}'_3$ and $\phi_{\widetilde{Q}'_3,3} \equiv 1$ on $\ell(\widetilde{Q}')[-\frac{15}{16}, -\frac{7}{48}]$.

On the other hand, observe that there exist bump functions $\phi_{Q''_i,i}$ ($i = 1, 2$) adapted to the shifted dyadic interval Q''_i such that $\text{supp } \phi_{Q''_i,i} \subseteq \frac{9}{10}Q''_i$ and $\phi_{Q''_i,i} \equiv 1$ on $\frac{8}{10}Q''_i$ ($i = 1, 2$) respectively, and $\text{supp } \phi_{Q''} \subseteq \frac{8}{10}Q''$, thus one has $\phi_{Q''_{1,1}} \cdot \phi_{Q''_{2,2}} \equiv 1$ on $\text{supp } \phi_{Q''}$. Since $\xi_1^2 \in \text{supp } \phi_{Q''_{1,1}} \subseteq \frac{9}{10}Q''_1$ and $\xi_2^2 \in \text{supp } \phi_{Q''_{2,2}} \subseteq \frac{9}{10}Q''_2$, it follows that $-\xi_1^2 - \xi_2^2 \in -\frac{9}{10}Q''_1 - \frac{9}{10}Q''_2$, and as a consequence, one can find a shifted dyadic interval Q''_3 with the property that $-\frac{9}{10}Q''_1 - \frac{9}{10}Q''_2 \subseteq \frac{7}{10}Q''_3$ and also satisfying $|Q''_1| = |Q''_2| \simeq |Q''_3|$. In particular, there exists bump function $\phi_{Q''_3,3}$ adapted to Q''_3 and supported in $\frac{9}{10}Q''_3$ such that $\phi_{Q''_3,3} \equiv 1$ on $-\frac{9}{10}Q''_1 - \frac{9}{10}Q''_2$.

We denote by $\widetilde{\mathbf{Q}}'$ the collection of all cubes $\widetilde{Q}' := \widetilde{Q}'_1 \times \widetilde{Q}'_2 \times \widetilde{Q}'_3$ with $\widetilde{Q}'_1 \times \widetilde{Q}'_2 \in \widetilde{\mathbb{Q}}'$ and \widetilde{Q}'_3 be defined as above, and denote by \mathbf{Q}'' the collection of all shifted dyadic quasi-cubes $Q'' := Q''_1 \times Q''_2 \times Q''_3$ with $Q''_1 \times Q''_2 \in \mathbb{Q}_{\mathbb{I}}''$ and Q''_3 be defined as above.

Definition 2.2. ([28]) We say that a collection of shifted dyadic quasi-cubes (cubes) is *sparse* if and only if for every $j = 1, 2, 3$,

- (i) whenever Q and \widetilde{Q} belong to this collection and $|Q_j| < |\widetilde{Q}_j|$ then $10^8|Q_j| \leq |\widetilde{Q}_j|$;
- (ii) whenever Q and \widetilde{Q} belong to this collection and $|Q_j| = |\widetilde{Q}_j|$ then $10^8Q_j \cap 10^8\widetilde{Q}_j = \emptyset$.

In fact, it is not difficult to see that the collection \mathbf{Q}'' can be split into a sum of finitely many *sparse* collection of shifted dyadic quasi-cubes. Therefore, we can assume from now on that the collection \mathbf{Q}'' is *sparse*.

Assuming this we then observe that, for any Q'' in such a sparse collection \mathbf{Q}'' , there exists a unique shifted dyadic cube \widetilde{Q}'' in \mathbb{R}^3 such that $Q'' \subseteq \frac{7}{10}\widetilde{Q}''$ and with property that $\text{diam}(Q'') \simeq \text{diam}(\widetilde{Q}'')$. This allows us in particular to assume further that \mathbf{Q}'' is a sparse collection of shifted dyadic cubes (that is, $|Q_1''| = |Q_2''| = |Q_3''| = \ell(Q'')$).

Now consider the trilinear form $\Lambda_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3)$ associated to $T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2)$, which can be written as

$$(2.17) \quad \begin{aligned} \Lambda_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3) &:= \int_{\mathbb{R}^2} T_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2)(x) f_3(x) dx \\ &= \sum_{\widetilde{Q}' \in \widetilde{\mathbf{Q}'}, Q'' \in \mathbf{Q}''} \int_{\xi_1 + \xi_2 + \xi_3 = 0} m_{\widetilde{Q}', Q''}^\varepsilon(\xi_1, \xi_2, \xi_3) (f_1 * (\check{\phi}_{\widetilde{Q}'_1, 1} \otimes \check{\phi}_{Q''_1, 1}))^\wedge(\xi_1) \\ &\quad \times (f_2 * (\check{\phi}_{\widetilde{Q}'_2, 2} \otimes \check{\phi}_{Q''_2, 2}))^\wedge(\xi_2) (f_3 * (\check{\phi}_{\widetilde{Q}'_3, 3} \otimes \check{\phi}_{Q''_3, 3}))^\wedge(\xi_3) d\xi_1 d\xi_2 d\xi_3, \end{aligned}$$

where $\xi_i = (\xi_i^1, \xi_i^2)$ for $i = 1, 2, 3$, while

$$(2.18) \quad m_{\widetilde{Q}', Q''}^\varepsilon(\xi_1, \xi_2, \xi_3) := m^\varepsilon(\xi_1, \xi_2) \cdot (\widetilde{\phi}_{\widetilde{Q}'} \otimes (\phi_{Q''_1 \times Q''_2} \cdot \widetilde{\phi}_{Q''_3, 3}))(\xi_1, \xi_2, \xi_3),$$

where $\widetilde{\phi}_{\widetilde{Q}'}$ is an appropriate smooth function of variable $(\xi_1^1, \xi_2^1, \xi_3^1)$ which is supported on a slightly larger cube (with a constant magnification independent of $\ell(\widetilde{Q}')$) than $\text{supp}(\phi_{\widetilde{Q}'_1, 1}(\xi_1^1)\phi_{\widetilde{Q}'_2, 2}(\xi_2^1)\phi_{\widetilde{Q}'_3, 3}(\xi_3^1))$ and equals 1 on $\text{supp}(\phi_{\widetilde{Q}'_1, 1}(\xi_1^1)\phi_{\widetilde{Q}'_2, 2}(\xi_2^1)\phi_{\widetilde{Q}'_3, 3}(\xi_3^1))$, the function $\phi_{Q''_1 \times Q''_2}(\xi_1^2, \xi_2^2)$ is one term of the partition of unity defined in (2.14), $\widetilde{\phi}_{Q''_3, 3}$ is an appropriate smooth function of variable ξ_3^2 supported on a slightly larger interval (with a constant magnification independent of $\ell(Q'')$) than $\text{supp} \phi_{Q''_3, 3}$, which equals 1 on $\text{supp} \phi_{Q''_3, 3}$. We can decompose $m_{\widetilde{Q}', Q''}^\varepsilon(\xi_1, \xi_2, \xi_3)$ as a Fourier series:

$$(2.19) \quad m_{\widetilde{Q}', Q''}^\varepsilon(\xi_1, \xi_2, \xi_3) = \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2} C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''} e^{2\pi i(n'_1, n'_2, n'_3) \cdot (\xi_1^1, \xi_2^1, \xi_3^1)/\ell(\widetilde{Q}')} e^{2\pi i(n''_1, n''_2, n''_3) \cdot (\xi_1^2, \xi_2^2, \xi_3^2)/\ell(Q'')},$$

where the Fourier coefficients $C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''}$ are given by

$$(2.20) \quad \begin{aligned} C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''} &= \int_{\mathbb{R}^6} m_{\widetilde{Q}', Q''}^\varepsilon((\ell(\widetilde{Q}')\xi_1^1, \ell(Q'')\xi_1^2), (\ell(\widetilde{Q}')\xi_2^1, \ell(Q'')\xi_2^2), (\ell(\widetilde{Q}')\xi_3^1, \ell(Q'')\xi_3^2)) \\ &\quad \times e^{-2\pi i(\vec{n}_1 \cdot \xi_1 + \vec{n}_2 \cdot \xi_2 + \vec{n}_3 \cdot \xi_3)} d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Then, by a straightforward calculation, we can rewrite (2.17) as

$$(2.21) \quad \begin{aligned} \Lambda_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3) &= \sum_{\widetilde{Q}' \in \widetilde{\mathbf{Q}'}, Q'' \in \mathbf{Q}''} \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2} C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''} \int_{\mathbb{R}^2} \\ &\quad (f_1 * (\check{\phi}_{\widetilde{Q}'_1, 1} \otimes \check{\phi}_{Q''_1, 1}))(x - (\frac{n'_1}{\ell(\widetilde{Q}')}, \frac{n''_1}{\ell(Q'')})) (f_2 * (\check{\phi}_{\widetilde{Q}'_2, 2} \otimes \check{\phi}_{Q''_2, 2}))(x - (\frac{n'_2}{\ell(\widetilde{Q}')}, \frac{n''_2}{\ell(Q'')})) \\ &\quad \times (f_3 * (\check{\phi}_{\widetilde{Q}'_3, 3} \otimes \check{\phi}_{Q''_3, 3}))(x - (\frac{n'_3}{\ell(\widetilde{Q}')}, \frac{n''_3}{\ell(Q'')})) dx. \end{aligned}$$

Definition 2.3. ([28, 32]) An arbitrary dyadic rectangle of area 1 in the phase-space plane is called a *Heisenberg box* or *tile*. Let $P := I_P \times \omega_P$ be a tile. A L^2 -normalized wave packet on P is a function Φ_P which has Fourier support $\text{supp } \hat{\Phi}_P \subseteq \frac{9}{10}\omega_P$ and obeys the estimates

$$|\Phi_P(x)| \lesssim |I_P|^{-\frac{1}{2}} \left(1 + \frac{\text{dist}(x, I_P)}{|I_P|}\right)^{-M}$$

for all $M > 0$, where the implicit constant depends on M .

Now we define $\phi_{\widetilde{Q}'_i, i}^{n'_i} := e^{2\pi i n'_i \xi_i^1 / \ell(\widetilde{Q}')} \cdot \phi_{\widetilde{Q}'_i, i}$ and $\phi_{Q''_i, i}^{n''_i} := e^{2\pi i n''_i \xi_i^2 / \ell(Q'')} \cdot \phi_{Q''_i, i}$ for $i = 1, 2, 3$.

Since any $\widetilde{Q}' \in \widetilde{\mathbf{Q}}'$ and $Q'' \in \mathbf{Q}''$ are both shifted dyadic cubes, there exists integers $k', k'' \in \mathbb{Z}$ such that $\ell(\widetilde{Q}') = |\widetilde{Q}'_1| = |\widetilde{Q}'_2| = |\widetilde{Q}'_3| = 2^{k'}$ and $\ell(Q'') = |Q''_1| = |Q''_2| = |Q''_3| = 2^{k''}$ respectively. By splitting the integral region \mathbb{R}^2 into the union of unit squares, the L^2 -normalization procedure and simple calculations, we can rewrite (2.21) as

$$\begin{aligned} (2.22) \quad & \Lambda_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3) \\ &= \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2} \sum_{\widetilde{Q}' \in \widetilde{\mathbf{Q}}', Q'' \in \mathbf{Q}''} \int_0^1 \int_0^1 \sum_{\substack{\widetilde{I}' \text{ dyadic,} \\ |\widetilde{I}'| = 2^{-k'}}} \sum_{\substack{I'' \text{ dyadic,} \\ |I''| = 2^{-k''}}} \frac{C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''}}{|\widetilde{I}'|^{\frac{1}{2}} \times |I''|^{\frac{1}{2}}} \langle f_1, \check{\phi}_{\widetilde{I}', \widetilde{Q}'_1, 1}^{n'_1, \nu'} \otimes \check{\phi}_{I'', Q''_1, 1}^{n''_1, \nu''} \rangle \\ & \quad \times \langle f_2, \check{\phi}_{\widetilde{I}', \widetilde{Q}'_2, 2}^{n'_2, \nu'} \otimes \check{\phi}_{I'', Q''_2, 2}^{n''_2, \nu''} \rangle \langle f_3, \check{\phi}_{\widetilde{I}', \widetilde{Q}'_3, 3}^{n'_3, \nu'} \otimes \check{\phi}_{I'', Q''_3, 3}^{n''_3, \nu''} \rangle d\nu' d\nu'' \\ &=: \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2} \int_0^1 \int_0^1 \sum_{\vec{P} := \widetilde{P}' \otimes P'' \in \widetilde{\mathbb{P}}} \frac{C_{\vec{Q}_{\vec{P}}, \vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \vec{n}_1, \nu} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \vec{n}_2, \nu} \rangle \langle f_3, \Phi_{\vec{P}_3}^{3, \vec{n}_3, \nu} \rangle d\nu, \end{aligned}$$

where the notation $\langle \cdot, \cdot \rangle$ denotes the complex scalar L^2 inner product, the Fourier coefficients $C_{\vec{Q}_{\vec{P}}, \vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon} := C_{\vec{n}_1, \vec{n}_2, \vec{n}_3}^{\varepsilon, \widetilde{Q}', Q''}$, the *tri-tiles* $\widetilde{P}' := (\widetilde{P}'_1, \widetilde{P}'_2, \widetilde{P}'_3)$ and $P'' := (P''_1, P''_2, P''_3)$, the *tiles* $\widetilde{P}'_i := I_{\widetilde{P}'_i} \times \omega_{\widetilde{P}'_i}$ with $I_{\widetilde{P}'_i} := \widetilde{I}' = 2^{-k'}[l', l' + 1] =: I_{\widetilde{P}'}$ and the frequency intervals $\omega_{\widetilde{P}'_i} := \widetilde{Q}'_i$ for $i = 1, 2, 3$, the *tiles* $P''_j := I_{P''_j} \times \omega_{P''_j}$ with $I_{P''_j} := I'' = 2^{-k''}[l'', l'' + 1] =: I_{P''}$ and the frequency intervals $\omega_{P''_j} := Q''_j$ for $j = 1, 2, 3$, the frequency cubes $Q_{\widetilde{P}'} := \omega_{\widetilde{P}'_1} \times \omega_{\widetilde{P}'_2} \times \omega_{\widetilde{P}'_3}$ and $Q_{P''} := \omega_{P''_1} \times \omega_{P''_2} \times \omega_{P''_3}$, $\widetilde{\mathbb{P}}'$ denotes a collection of such tri-tiles \widetilde{P}' and \mathbb{P}'' denotes a collection of such tri-tiles P'' , the bi-tiles \vec{P}_1, \vec{P}_2 and \vec{P}_3 are defined by

$$\begin{aligned} \vec{P}_1 &:= (\widetilde{P}'_1, P''_1) = (2^{-k'}[l', l' + 1] \times 2^{k'}[-\frac{1}{2}, \frac{1}{2}], 2^{-k''}[l'', l'' + 1] \times Q''_1), \\ \vec{P}_2 &:= (\widetilde{P}'_2, P''_2) = (2^{-k'}[l', l' + 1] \times 2^{k'}[\frac{1}{24}, \frac{25}{24}], 2^{-k''}[l'', l'' + 1] \times Q''_2), \\ \vec{P}_3 &:= (\widetilde{P}'_3, P''_3) = (2^{-k'}[l', l' + 1] \times 2^{k'}[-\frac{25}{24}, -\frac{1}{24}], 2^{-k''}[l'', l'' + 1] \times Q''_3); \end{aligned}$$

the bi-parameter tri-tile $\vec{P} := \widetilde{P}' \otimes P'' = (\vec{P}_1, \vec{P}_2, \vec{P}_3)$, the rectangles $I_{\vec{P}_i} := I_{\widetilde{P}'_i} \times I_{P''_i} = I_{\widetilde{P}'} \times I_{P''} =: I_{\vec{P}}$ for $i = 1, 2, 3$ and hence $|I_{\vec{P}}| = |I_{\widetilde{P}'} \times I_{P''}| = |I_{\vec{P}_1}| = |I_{\vec{P}_2}| = |I_{\vec{P}_3}| = 2^{-k'} \cdot 2^{-k''}$, the double frequency cube $Q_{\vec{P}} := (Q_{\widetilde{P}'}, Q_{P''}) = (\omega_{\widetilde{P}'_1} \times \omega_{\widetilde{P}'_2} \times \omega_{\widetilde{P}'_3}, \omega_{P''_1} \times \omega_{P''_2} \times \omega_{P''_3})$, $\vec{\mathbb{P}} := \widetilde{\mathbb{P}}' \times \mathbb{P}''$ denotes a collection of such bi-parameter tri-tiles \vec{P} ; while

the L^2 -normalized wave packets $\Phi_{\widetilde{P}'_i}^{i,n'_i,\nu'}$ associated with the Heisenberg boxes \widetilde{P}'_i are defined by $\Phi_{\widetilde{P}'_i}^{i,n'_i,\nu'}(x_1) := \check{\phi}_{\widetilde{I}',\widetilde{Q}'_i,i}^{n'_i,\nu'}(x_1) := 2^{-\frac{k'}{2}} \overline{\check{\phi}_{\widetilde{Q}'_i,i}^{n'_i}}(2^{-k'}(l' + \nu') - x_1)$ for $i = 1, 2, 3$, the L^2 -normalized wave packets $\Phi_{P''_i}^{i,n''_i,\nu''}$ associated with the Heisenberg boxes P''_i are defined by $\Phi_{P''_i}^{i,n''_i,\nu''}(x_2) := \check{\phi}_{I'',Q''_i,i}^{n''_i,\nu''}(x_2) := 2^{-\frac{k''}{2}} \overline{\check{\phi}_{Q''_i,i}^{n''_i}}(2^{-k''}(l'' + \nu'') - x_2)$ for $i = 1, 2, 3$, the smooth bump functions $\Phi_{\widetilde{P}_i}^{i,\vec{n}_i,\nu} := \Phi_{\widetilde{P}'_i}^{i,n'_i,\nu'} \otimes \Phi_{P''_i}^{i,n''_i,\nu''}$ for $i = 1, 2, 3$.

We have the following rapid decay estimates of the Fourier coefficients $C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}^\varepsilon$ with respect to the parameters $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$.

Lemma 2.4. *The Fourier coefficients $C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}^\varepsilon$ satisfy estimates*

$$(2.23) \quad |C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}^\varepsilon| \lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{n}_j|)^M} \cdot C_{|I_{\widetilde{P}'}|}^\varepsilon$$

for any bi-parameter tri-tile $\vec{P} \in \vec{\mathbb{P}}$, where M is sufficiently large and the sequence $C_k^\varepsilon := C_{|I_{\widetilde{P}'}|}^\varepsilon$ for $|I_{\widetilde{P}'}| = 2^{-k'}$ ($k' \in \mathbb{Z}$) satisfies

$$(2.24) \quad \sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon \leq C_\varepsilon < +\infty$$

and $C_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

Proof. Let $\ell(Q_{\widetilde{P}'}) = 2^{k'}$ and $\ell(Q_{P''}) = 2^{k''}$ for $k', k'' \in \mathbb{Z}$. For any $\varepsilon > 0$, $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$ and $\vec{P} \in \vec{\mathbb{P}}$, we deduce from (2.18) and (2.20) that

$$(2.25) \quad \begin{aligned} C_{Q_{\vec{P}},\vec{n}_1,\vec{n}_2,\vec{n}_3}^\varepsilon &= \int_{\mathbb{R}^6} m_{Q_{\widetilde{P}'},Q_{P''}}^\varepsilon((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) \\ &\quad \times e^{-2\pi i(\vec{n}_1 \cdot \xi_1 + \vec{n}_2 \cdot \xi_2 + \vec{n}_3 \cdot \xi_3)} d\xi_1 d\xi_2 d\xi_3, \end{aligned}$$

where

$$(2.26) \quad \begin{aligned} m_{Q_{\widetilde{P}'},Q_{P''}}^\varepsilon((2^{k'}\xi_1^1, 2^{k''}\xi_1^2), (2^{k'}\xi_2^1, 2^{k''}\xi_2^2), (2^{k'}\xi_3^1, 2^{k''}\xi_3^2)) &:= m^\varepsilon(2^{k'}\bar{\xi}_1, 2^{k''}\bar{\xi}_2) \\ &\quad \times \widetilde{\phi}_{Q_{\widetilde{P}'}}(2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1) \phi_{\omega_{P'_1} \times \omega_{P'_2}}(2^{k''}\bar{\xi}_2) \widetilde{\phi}_{\omega_{P'_3},3}(2^{k''}\xi_3^2). \end{aligned}$$

Observe that $\text{supp}(\widetilde{\phi}_{Q_{\widetilde{P}'}}(\xi_1^1, \xi_2^1, \xi_3^1) \phi_{\omega_{P'_1} \times \omega_{P'_2}}(\bar{\xi}_2) \widetilde{\phi}_{\omega_{P'_3},3}(\xi_3^2)) \subseteq Q_{\widetilde{P}'} \times Q_{P''}$, we have that $\text{supp}(\widetilde{\phi}_{Q_{\widetilde{P}'}}(2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1) \phi_{\omega_{P'_1} \times \omega_{P'_2}}(2^{k''}\bar{\xi}_2) \widetilde{\phi}_{\omega_{P'_3},3}(2^{k''}\xi_3^2)) \subseteq Q_{\widetilde{P}'}^0 \times Q_{P''}^0$, where cubes $Q_{\widetilde{P}'}^0$ and $Q_{P''}^0$ are defined by

$$(2.27) \quad Q_{\widetilde{P}'}^0 = \omega_{P'_1}^0 \times \omega_{P'_2}^0 \times \omega_{P'_3}^0 := \{(\xi_1^1, \xi_2^1, \xi_3^1) \in \mathbb{R}^3 : (2^{k'}\xi_1^1, 2^{k'}\xi_2^1, 2^{k'}\xi_3^1) \in Q_{\widetilde{P}'}\},$$

$$(2.28) \quad Q_{P''}^0 = \omega_{P''_1}^0 \times \omega_{P''_2}^0 \times \omega_{P''_3}^0 := \{(\xi_1^2, \xi_2^2, \xi_3^2) \in \mathbb{R}^3 : (2^{k''}\xi_1^2, 2^{k''}\xi_2^2, 2^{k''}\xi_3^2) \in Q_{P''}\}$$

and satisfy $|Q_{\widetilde{P}'}^0| \simeq |Q_{P''}^0| \simeq 1$. From the properties of the *Whitney squares* we constructed above, one obtains that $\text{dist}(2^{k'}\bar{\xi}_1, \Gamma_1) \simeq 2^{k'}$ for any $\bar{\xi}_1 \in \omega_{P'_1}^0 \times \omega_{P'_2}^0$ and $\text{dist}(2^{k''}\bar{\xi}_2, \Gamma_2) \simeq 2^{k''}$ for any $\bar{\xi}_2 \in \omega_{P''_1}^0 \times \omega_{P''_2}^0$.

One can deduce from (2.25), (2.26) and integrating by parts sufficiently many times that

$$\begin{aligned}
|C_{Q_{\vec{P}}, \vec{n}_1, \vec{n}_2, \vec{n}_3}^\varepsilon| &\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{n}_j|)^M} \\
&\times \int_{Q_{\vec{P}'}^0 \times Q_{P''}^0} |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} [m_{Q_{\vec{P}'}, Q_{P''}}^\varepsilon((2^{k'} \xi_1^1, 2^{k''} \xi_1^2), (2^{k'} \xi_2^1, 2^{k''} \xi_2^2), (2^{k'} \xi_3^1, 2^{k''} \xi_3^2))]| d\xi_1 d\xi_2 d\xi_3 \\
&\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{n}_j|)^M} \int_{\omega_{P_1''}^0 \times \omega_{P_2''}^0} \text{dist}(2^{k''} \bar{\xi}_2, \Gamma_2)^{|\alpha''|} \\
&\quad \times \int_{\omega_{P_1'}^0 \times \omega_{P_2'}^0} \text{dist}(2^{k'} \bar{\xi}_1, \Gamma_1)^{|\alpha'|} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} m^\varepsilon(2^{k'} \bar{\xi}_1, 2^{k''} \bar{\xi}_2)| d\bar{\xi}_1 d\bar{\xi}_2 \\
&\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{n}_j|)^M} \cdot \frac{1}{\ell(Q_{P''})^2} \int_{\omega_{P_1''} \times \omega_{P_2''}} \text{dist}(\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \\
&\quad \times \int_{\omega_{\widetilde{P}_1'} \times \omega_{\widetilde{P}_2'}} \text{dist}(\bar{\xi}_1, \Gamma_1)^{|\alpha'| - 2} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} m^\varepsilon(\bar{\xi}_1, \bar{\xi}_2)| d\bar{\xi}_1 d\bar{\xi}_2 =: \prod_{j=1}^3 \frac{1}{(1 + |\vec{n}_j|)^M} \cdot C_{|\widetilde{P}_j'|}^\varepsilon,
\end{aligned}$$

where the multi-indices $\alpha_i := (\alpha_i^1, \alpha_i^2)$ for $i = 1, 2, 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3| = M$ are sufficiently large, the multi-indices $\alpha' := (\alpha_1', \alpha_2', \alpha_3')$, $\alpha'' := (\alpha_1'', \alpha_2'', \alpha_3'')$ with $\alpha_i' \leq \alpha_i^1$ and $\alpha_j'' \leq \alpha_j^2$ for $i, j = 1, 2, 3$. This proves the estimates (2.23).

Moreover, for $|I_{\widetilde{P}'}| = 2^{-k'}$, we define the sequence $C_{k'}^\varepsilon := C_{|I_{\widetilde{P}'}|}^\varepsilon$ ($k' \in \mathbb{Z}$). From the estimates (1.10) for symbol $m^\varepsilon(\bar{\xi}_1, \bar{\xi}_2)$, we get that

$$(2.29) \quad \text{dist}(\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \cdot \int_{\mathbb{R}^2} \text{dist}(\bar{\xi}_1, \Gamma_1)^{|\alpha'| - 2} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} m^\varepsilon(\bar{\xi})| d\bar{\xi}_1 \leq B(\varepsilon) < +\infty,$$

and hence we can deduce the following summable property for the sequence $\{C_{k'}^\varepsilon\}_{k' \in \mathbb{Z}}$:

$$\begin{aligned}
\sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon &\lesssim \frac{1}{\ell(Q_{P''})^2} \int_{\omega_{P_1''} \times \omega_{P_2''}} \text{dist}(\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \\
(2.30) \quad &\times \int_{\cup_{\widetilde{P}' \in \widetilde{\mathbb{P}'}} (\omega_{\widetilde{P}_1'} \times \omega_{\widetilde{P}_2'})_{\widetilde{P}'}} \text{dist}(\bar{\xi}_1, \Gamma_1)^{|\alpha'| - 2} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} m^\varepsilon(\bar{\xi}_1, \bar{\xi}_2)| d\bar{\xi}_1 d\bar{\xi}_2 \\
&\lesssim \frac{1}{\ell(Q_{P''})^2} \int_{\omega_{P_1''} \times \omega_{P_2''}} B(\varepsilon) d\bar{\xi}_2 \leq C_\varepsilon < +\infty
\end{aligned}$$

and $C_\varepsilon \sim B(\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, this ends the proof of the summable estimate (2.24). \square

Observe that the rapid decay with respect to the parameters $\vec{n}_1, \vec{n}_2, \vec{n}_3 \in \mathbb{Z}^2$ in (2.23) is acceptable for summation, all the functions $\Phi_{\widetilde{P}_i'}^{i, n_i', \nu'}$ ($i = 1, 2, 3$) are L^2 normalized and are wave packets associated with the Heisenberg boxes \widetilde{P}_i' uniformly with respect to the parameters n_i' and all the functions $\Phi_{P_j''}^{j, n_j'', \nu''}$ ($j = 1, 2, 3$) are L^2 normalized and

are wave packets associated with the Heisenberg boxes P_j'' uniformly with respect to the parameters n_j'' , therefore we only need to consider from now on the part of the trilinear form $\Lambda_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3)$ defined in (2.22) corresponding to $\vec{n}_1 = \vec{n}_2 = \vec{n}_3 = \vec{0}$:

$$(2.31) \quad \dot{\Lambda}_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3) := \int_0^1 \int_0^1 \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \nu} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \nu} \rangle \langle f_3, \Phi_{\vec{P}_3}^{3, \nu} \rangle d\nu,$$

where $C_{Q_{\vec{P}}}^\varepsilon := C_{Q_{\vec{P}}, \vec{0}, \vec{0}, \vec{0}}^\varepsilon$, parameters $\nu = (\nu', \nu'')$ and $\Phi_{\vec{P}_i}^{i, \nu} := \Phi_{\vec{P}_i}^{i, \vec{0}, \nu}$ for $i = 1, 2, 3$.

Remark 2.5. We should point out two important properties of the tri-tiles in \mathbb{P}'' (see [25, 28]). First, if one knows the position of P_1'' , P_2'' or P_3'' , then one knows precisely the positions of the other two as well. Second, if one assumes for instance that all the frequency intervals $\omega_{P_1''}$ of the P_1'' tiles intersect each other (say, they are non-lacunary about a fixed frequency ξ_0), then the frequency intervals $\omega_{P_2''}$ of the corresponding P_2'' tiles are disjoint and lacunary around ξ_0 (that is, $\text{dist}(\xi_0, \omega_{P_2''}) \simeq |\omega_{P_2''}|$ for all $P_2'' \in \mathbb{P}''$). A similar conclusion can also be drawn for the P_3'' tiles modulo certain translations. This observation motivates the introduction of *trees* in Definition 3.1.

We review the following definitions from [28].

Definition 2.6. A collection \mathbb{P} of tri-tiles is called *sparse*, if all tri-tiles in \mathbb{P} have the same shift and the sets $\{Q_P : P \in \mathbb{P}\}$ and $\{I_P : P \in \mathbb{P}\}$ are sparse.

Definition 2.7. Let P and P' be tiles. Then

- (i) we write $P' < P$ if $I_{P'} \subsetneq I_P$ and $\omega_P \subseteq 3\omega_{P'}$;
- (ii) we write $P' \leq P$ if $P' < P$ or $P' = P$;
- (iii) we write $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $\omega_P \subseteq 10^6 \omega_{P'}$;
- (iv) we write $P' \lesssim' P$ if $P' \lesssim P$ but $P' \not\leq P$.

Definition 2.8. A collection \mathbb{P} of tri-tiles is said to have *rank 1* if the following properties are satisfied for all $P, P' \in \mathbb{P}$.

- (i) If $P \neq P'$, then $P_j \neq P'_j$ for $1 \leq j \leq 3$.
- (ii) If $\omega_{P_j} = \omega_{P'_j}$ for some j , then $\omega_{P_j} = \omega_{P'_j}$ for all $1 \leq j \leq 3$.
- (iii) If $P'_j \leq P_j$ for some j , then $P'_j \lesssim P_j$ for all $1 \leq j \leq 3$.
- (iv) If in addition to $P'_j \leq P_j$ one also assumes that $10^8 |I_{P'}| \leq |I_P|$, then one has $P'_i \lesssim' P_i$ for every $i \neq j$.

It is not difficult to observe that the collection of tri-tiles \mathbb{P}'' can be written as a finite union of sparse collections of rank 1, thus we may assume further that \mathbb{P}'' is a sparse collection of rank 1 from now on.

The bilinear operator corresponding to the trilinear form $\dot{\Lambda}_{m^\varepsilon, (lh, \mathbb{I})}^{(2)}(f_1, f_2, f_3)$ can be written as

$$(2.32) \quad \dot{\Pi}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2)(x) = \int_0^1 \int_0^1 \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \nu} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \nu} \rangle \Phi_{\vec{P}_3}^{3, \nu}(x) d\nu.$$

Since $\dot{\Pi}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2)$ is an average of some discrete bilinear model operators depending on the parameters $\nu = (\nu_1, \nu_2) \in [0, 1]^2$, it is enough to prove the Hölder-type L^p estimates for

each of them, uniformly with respect to parameters $\nu = (\nu_1, \nu_2)$. From now on, we will do this in the particular case when the parameters $\nu = (\nu_1, \nu_2) = (0, 0)$, but the same argument works in general. By Fatou's lemma, we can also restrict the summation in the definition (2.32) of $\dot{\Pi}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2)$ on collection $\vec{\mathbb{P}} = \tilde{\mathbb{P}}' \times \mathbb{P}''$ with arbitrary finite collections $\tilde{\mathbb{P}}'$ and \mathbb{P}'' of tri-tiles, and prove the estimates are uniform with respect to different choices of the set $\vec{\mathbb{P}}$.

Therefore, one can reduce the bilinear operator $\dot{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ further to the discrete bilinear model operator $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ defined by

$$(2.33) \quad \Pi_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2)(x) := \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{C_{\vec{P}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^1 \rangle \langle f_2, \Phi_{\vec{P}_2}^2 \rangle \Phi_{\vec{P}_3}^3(x),$$

where $\Phi_{\vec{P}_j}^j := \Phi_{\vec{P}_j}^{j,(0,0)}$ for $j = 1, 2, 3$ respectively, $\vec{\mathbb{P}} = \tilde{\mathbb{P}}' \times \mathbb{P}''$ with arbitrary finite collection $\tilde{\mathbb{P}}'$ of tri-tiles and arbitrary finite sparse collection \mathbb{P}'' of rank 1. As have discussed above, we now reach a conclusion that the proof of Theorem 1.3 can be reduced to proving the following L^p estimates for discrete bilinear model operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$.

Proposition 2.9. *If the finite set $\vec{\mathbb{P}}$ is chosen arbitrarily as above, then the operator $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ given by (2.33) maps $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ boundedly for any $1 < p_1, p_2 \leq \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. Moreover, the implicit constants in the bounds depend only on ε, p_1, p_2, p and are independent of the particular choice of finite collection $\vec{\mathbb{P}}$.*

2.1.2. Discretization for bilinear, bi-parameter operators $T_{\tilde{m}^\varepsilon}^{(2)}$. We will proceed the discretization procedure as follows. First, we need to decompose the symbol $\tilde{m}^\varepsilon(\xi)$ in a natural way. To this end, for both the spatial variables x_i ($i = 1, 2$), we decompose the regions $\{\bar{\xi}_i = (\xi_1^i, \xi_2^i) \in \mathbb{R}^2 : \xi_1^i \neq \xi_2^i\}$ by using *Whitney squares* with respect to the singularity lines $\Gamma_i = \{\xi_1^i = \xi_2^i\}$ ($i = 1, 2$) respectively. Since the *Whitney dyadic square* decomposition for the x_2 direction has already been described in (2.11), (2.12), (2.13) and (2.14) in sub-subsection 2.1.1, we only need to discuss the *Whitney* decomposition with respect to the singularity line Γ_1 in x_1 direction.

To be specific, we consider the collection \mathbb{Q}' of all shifted dyadic squares $Q' = Q'_1 \times Q'_2$ satisfying

$$(2.34) \quad Q' \subseteq \{(\xi_1^1, \xi_2^1) \in \mathbb{R}^2 : \xi_1^1 \neq \xi_2^1\}, \quad \text{dist}(Q', \Gamma_1) \simeq 10^4 \text{diam}(Q').$$

We can split the collection \mathbb{Q}' into two disjoint sub-collections, that is, define

$$(2.35) \quad \mathbb{Q}'_{\text{I}} := \{Q' \in \mathbb{Q}' : Q' \subseteq \{\xi_1^1 < \xi_2^1\}\}, \quad \mathbb{Q}'_{\text{II}} := \{Q' \in \mathbb{Q}' : Q' \subseteq \{\xi_1^1 > \xi_2^1\}\}.$$

Since the set of squares $\{\frac{7}{10}Q' : Q' \in \mathbb{Q}'\}$ also forms a finitely overlapping cover of the region $\{\xi_1^1 \neq \xi_2^1\}$, we can apply a standard partition of unity and write the symbol $\chi_{\{\xi_1^1 \neq \xi_2^1\}}$ as

$$(2.36) \quad \chi_{\{\xi_1^1 \neq \xi_2^1\}} = \sum_{Q' \in \mathbb{Q}'} \phi_{Q'}(\xi_1^1, \xi_2^1) = \left\{ \sum_{Q' \in \mathbb{Q}'_{\text{I}}} + \sum_{Q' \in \mathbb{Q}'_{\text{II}}} \right\} \phi_{Q'}(\xi_1^1, \xi_2^1) = \chi_{\{\xi_1^1 < \xi_2^1\}} + \chi_{\{\xi_1^1 > \xi_2^1\}},$$

where each $\phi_{Q'}$ is a smooth bump function adapted to Q' and supported in $\frac{8}{10}Q'$.

Notice that by splitting the symbol $\tilde{m}^\varepsilon(\xi)$, we can decompose the operator $T_{\tilde{m}^\varepsilon}^{(2)}$ correspondingly into a finite sum of several parts and we only need to discuss in detail arbitrary one of them. From the decompositions (2.13) and (2.36), we obtain that

$$\begin{aligned}
 \tilde{m}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) &= \left\{ \sum_{\substack{Q' \in \mathbb{Q}'_I, \\ Q'' \in \mathbb{Q}''_I}} + \sum_{\substack{Q' \in \mathbb{Q}'_I, \\ Q'' \in \mathbb{Q}''_{II}}} + \sum_{\substack{Q' \in \mathbb{Q}'_{II}, \\ Q'' \in \mathbb{Q}''_I}} + \sum_{\substack{Q' \in \mathbb{Q}'_{II}, \\ Q'' \in \mathbb{Q}''_{II}}} \right\} \phi_{Q'}(\xi_1^1, \xi_2^1) \phi_{Q''}(\xi_1^2, \xi_2^2) \cdot \tilde{m}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) \\
 (2.37) \quad &=: \tilde{m}_{I,I}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) + \tilde{m}_{I,II}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) + \tilde{m}_{II,I}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) + \tilde{m}_{II,II}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2).
 \end{aligned}$$

One can easily observe that we only need to discuss in detail one term in the decomposition (2.37), since the other term can be treated in the same way. Without loss of generality, we will only consider the third term in the decomposition (2.37), which can be written as

$$(2.38) \quad \tilde{m}_{II,I}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) := \sum_{Q' \in \mathbb{Q}'_{II}, Q'' \in \mathbb{Q}''_I} \tilde{m}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2) \phi_{Q'}(\xi_1^1, \xi_2^1) \phi_{Q''}(\xi_1^2, \xi_2^2).$$

In a word, we only need to consider the bilinear operator $T_{\tilde{m}_{II,I}^\varepsilon}^{(2)}$ given by

$$(2.39) \quad T_{\tilde{m}_{II,I}^\varepsilon}^{(2)}(f_1, f_2)(x) := \sum_{Q' \in \mathbb{Q}'_{II}, Q'' \in \mathbb{Q}''_I} \int_{\mathbb{R}^4} \tilde{m}^\varepsilon(\xi) \phi_{Q'}(\bar{\xi}_1) \phi_{Q''}(\bar{\xi}_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi$$

from now on, and the proof of Theorem 1.4 can be reduced to proving the following L^p estimates for $T_{\tilde{m}_{II,I}^\varepsilon}^{(2)}$:

$$(2.40) \quad \|T_{\tilde{m}_{II,I}^\varepsilon}^{(2)}(f_1, f_2)\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon, p, p_1, p_2} \|f_1\|_{L^{p_1}(\mathbb{R}^2)} \cdot \|f_2\|_{L^{p_2}(\mathbb{R}^2)},$$

as long as $1 < p_1, p_2 \leq \infty$ and $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$.

Observe that there exist bump functions $\phi_{Q'_i, i}$ ($i = 1, 2$) adapted to the shifted dyadic interval Q'_i such that $\text{supp } \phi_{Q'_i, i} \subseteq \frac{9}{10}Q'_i$ and $\phi_{Q'_i, i} \equiv 1$ on $\frac{8}{10}Q'_i$ ($i = 1, 2$) respectively, and $\text{supp } \phi_{Q'} \subseteq \frac{8}{10}Q'$, thus one has $\phi_{Q'_1, 1} \cdot \phi_{Q'_2, 2} \equiv 1$ on $\text{supp } \phi_{Q'}$. Since $\xi_1^1 \in \text{supp } \phi_{Q'_1, 1} \subseteq \frac{9}{10}Q'_1$ and $\xi_2^1 \in \text{supp } \phi_{Q'_2, 2} \subseteq \frac{9}{10}Q'_2$, it follows that $-\xi_1^1 - \xi_2^1 \in -\frac{9}{10}Q'_1 - \frac{9}{10}Q'_2$, and as a consequence, one can find a shifted dyadic interval Q'_3 with the property that $-\frac{9}{10}Q'_1 - \frac{9}{10}Q'_2 \subseteq \frac{7}{10}Q'_3$ and also satisfying $|Q'_1| = |Q'_2| \simeq |Q'_3|$. In particular, there exists bump function $\phi_{Q'_3, 3}$ adapted to Q'_3 and supported in $\frac{9}{10}Q'_3$ such that $\phi_{Q'_3, 3} \equiv 1$ on $-\frac{9}{10}Q'_1 - \frac{9}{10}Q'_2$. Recall that the smooth functions $\phi_{Q'_j, j}$ ($j = 1, 2, 3$) and shifted dyadic intervals Q'_3 have already been defined in sub-subsection 2.1.1.

We denote by \mathbf{Q}' the collection of all shifted dyadic quasi-cubes $Q' := Q'_1 \times Q'_2 \times Q'_3$ with $Q'_1 \times Q'_2 \in \mathbb{Q}'_{II}$ and Q'_3 be defined as above, and denote by \mathbf{Q}'' the collection of all shifted dyadic quasi-cubes $Q'' := Q''_1 \times Q''_2 \times Q''_3$ with $Q''_1 \times Q''_2 \in \mathbb{Q}''_I$ and Q''_3 be defined in sub-subsection 2.1.1.

In fact, it is not difficult to see that the collections \mathbf{Q}' and \mathbf{Q}'' can be split into a sum of finitely many *sparse* collection of shifted dyadic quasi-cubes. Therefore, we can assume from now on that the collections \mathbf{Q}' and \mathbf{Q}'' is *sparse*.

Assuming this we then observe that, for any Q' in such a *sparse* collection \mathbf{Q}' , there exists a unique shifted dyadic cube \widetilde{Q}' in \mathbb{R}^3 such that $Q' \subseteq \frac{7}{10}\widetilde{Q}'$ and with property that $\text{diam}(Q') \simeq \text{diam}(\widetilde{Q}')$. This allows us in particular to assume further that \mathbf{Q}' is a *sparse*

collection of shifted dyadic cubes (that is, $|Q'_1| = |Q'_2| = |Q'_3| = \ell(Q')$). Similarly, we can also assume that \mathbf{Q}'' is a sparse collection of shifted dyadic cubes.

Now consider the trilinear form $\Lambda_{\tilde{m}_{\Pi, \mathbf{I}}}^{(2)}(f_1, f_2, f_3)$ associated to $T_{\tilde{m}_{\Pi, \mathbf{I}}}^{(2)}(f_1, f_2)$, which can be written as

$$(2.41) \quad \begin{aligned} \Lambda_{\tilde{m}_{\Pi, \mathbf{I}}}^{(2)}(f_1, f_2, f_3) &:= \int_{\mathbb{R}^2} T_{\tilde{m}_{\Pi, \mathbf{I}}}^{(2)}(f_1, f_2)(x) f_3(x) dx \\ &= \sum_{Q' \in \mathbf{Q}', Q'' \in \mathbf{Q}''} \int_{\xi_1 + \xi_2 + \xi_3 = 0} \tilde{m}_{Q', Q''}^{\varepsilon}(\xi_1, \xi_2, \xi_3) (f_1 * (\check{\phi}_{Q'_1, 1} \otimes \check{\phi}_{Q''_1, 1}))^{\wedge}(\xi_1) \\ &\quad \times (f_2 * (\check{\phi}_{Q'_2, 2} \otimes \check{\phi}_{Q''_2, 2}))^{\wedge}(\xi_2) (f_3 * (\check{\phi}_{Q'_3, 3} \otimes \check{\phi}_{Q''_3, 3}))^{\wedge}(\xi_3) d\xi_1 d\xi_2 d\xi_3, \end{aligned}$$

where $\xi_i = (\xi_i^1, \xi_i^2)$ for $i = 1, 2, 3$, while

$$(2.42) \quad \tilde{m}_{Q', Q''}^{\varepsilon}(\xi_1, \xi_2, \xi_3) := \tilde{m}^{\varepsilon}(\xi_1, \xi_2) \cdot ((\phi_{Q'_1 \times Q'_2} \cdot \tilde{\phi}_{Q'_3, 3}) \otimes (\phi_{Q''_1 \times Q''_2} \cdot \tilde{\phi}_{Q''_3, 3}))(\xi_1, \xi_2, \xi_3),$$

where $\tilde{\phi}_{Q'_3, 3}$ is an appropriate smooth function of variable ξ_3^1 supported on a slightly larger interval (with a constant magnification independent of $\ell(Q')$) than $\text{supp } \phi_{Q'_3, 3}$, which equals 1 on $\text{supp } \phi_{Q'_3, 3}$, and $\tilde{\phi}_{Q''_3, 3}$ is an appropriate smooth function of variable ξ_3^2 supported on a slightly larger interval (with a constant magnification independent of $\ell(Q'')$) than $\text{supp } \phi_{Q''_3, 3}$, which equals 1 on $\text{supp } \phi_{Q''_3, 3}$. We can decompose $\tilde{m}_{Q', Q''}^{\varepsilon}(\xi_1, \xi_2, \xi_3)$ as a Fourier series:

$$(2.43) \quad \tilde{m}_{Q', Q''}^{\varepsilon}(\xi_1, \xi_2, \xi_3) = \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''} e^{2\pi i(\vec{l}'_1, \vec{l}'_2, \vec{l}'_3) \cdot (\xi_1^1, \xi_2^1, \xi_3^1)/\ell(Q')} e^{2\pi i(\vec{l}''_1, \vec{l}''_2, \vec{l}''_3) \cdot (\xi_1^2, \xi_2^2, \xi_3^2)/\ell(Q'')},$$

where the Fourier coefficients $\tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''}$ are given by

$$(2.44) \quad \begin{aligned} \tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''} &= \int_{\mathbb{R}^6} \tilde{m}_{Q', Q''}^{\varepsilon}((\ell(Q')\xi_1^1, \ell(Q'')\xi_1^2), (\ell(Q')\xi_2^1, \ell(Q'')\xi_2^2), (\ell(Q')\xi_3^1, \ell(Q'')\xi_3^2)) \\ &\quad \times e^{-2\pi i(\vec{l}_1 \cdot \xi_1 + \vec{l}_2 \cdot \xi_2 + \vec{l}_3 \cdot \xi_3)} d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Then, by a straightforward calculation, we can rewrite (2.41) as

$$(2.45) \quad \begin{aligned} \Lambda_{\tilde{m}_{\Pi, \mathbf{I}}}^{(2)}(f_1, f_2, f_3) &= \sum_{Q' \in \mathbf{Q}', Q'' \in \mathbf{Q}''} \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''} \int_{\mathbb{R}^2} \\ &\quad (f_1 * (\check{\phi}_{Q'_1, 1} \otimes \check{\phi}_{Q''_1, 1}))(x - (\frac{l'_1}{\ell(Q')}, \frac{l''_1}{\ell(Q'')})) (f_2 * (\check{\phi}_{Q'_2, 2} \otimes \check{\phi}_{Q''_2, 2}))(x - (\frac{l'_2}{\ell(Q')}, \frac{l''_2}{\ell(Q'')})) \\ &\quad \times (f_3 * (\check{\phi}_{Q'_3, 3} \otimes \check{\phi}_{Q''_3, 3}))(x - (\frac{l'_3}{\ell(Q')}, \frac{l''_3}{\ell(Q'')})) dx. \end{aligned}$$

Now we define $\phi_{Q'_i, i}^{l'_i} := e^{2\pi i l'_i \xi_i^1 / \ell(Q')} \cdot \phi_{Q'_i, i}$ and $\phi_{Q''_i, i}^{l''_i} := e^{2\pi i l''_i \xi_i^2 / \ell(Q'')} \cdot \phi_{Q''_i, i}$ for $i = 1, 2, 3$. Since any $Q' \in \mathbf{Q}'$ and $Q'' \in \mathbf{Q}''$ are both shifted dyadic cubes, there exists integers $k', k'' \in \mathbb{Z}$ such that $\ell(Q') = |Q'_1| = |Q'_2| = |Q'_3| = 2^{k'}$ and $\ell(Q'') = |Q''_1| = |Q''_2| = |Q''_3| = 2^{k''}$ respectively. By splitting the integral region \mathbb{R}^2 into the union of unit squares, the

L^2 -normalization procedure and simple calculations, we can rewrite (2.45) as

$$\begin{aligned}
(2.46) \quad & \Lambda_{\tilde{m}_{\mathbb{H},\mathbb{I}}}^{(2)}(f_1, f_2, f_3) \\
&= \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \sum_{Q' \in \mathbf{Q}', Q'' \in \mathbf{Q}''} \int_0^1 \int_0^1 \sum_{\substack{I' \text{ dyadic,} \\ |I'|=2^{-k'}}} \sum_{\substack{I'' \text{ dyadic,} \\ |I''|=2^{-k''}}} \frac{\tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''}}{|I'|^{\frac{1}{2}} \times |I''|^{\frac{1}{2}}}} \langle f_1, \check{\phi}_{I', Q'_1, 1}^{l'_1, \lambda'} \otimes \check{\phi}_{I'', Q''_1, 1}^{l''_1, \lambda''} \rangle \\
&\quad \times \langle f_2, \check{\phi}_{I', Q'_2, 2}^{l'_2, \lambda'} \otimes \check{\phi}_{I'', Q''_2, 2}^{l''_2, \lambda''} \rangle \langle f_3, \check{\phi}_{I', Q'_3, 3}^{l'_3, \lambda'} \otimes \check{\phi}_{I'', Q''_3, 3}^{l''_3, \lambda''} \rangle d\lambda' d\lambda'' \\
&=: \sum_{\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2} \int_0^1 \int_0^1 \sum_{\vec{P} := P' \otimes P'' \in \tilde{\mathbb{P}}} \frac{\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon}}{|I_{\vec{P}}|^{\frac{1}{2}}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \vec{l}_1, \lambda} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \vec{l}_2, \lambda} \rangle \langle f_3, \Phi_{\vec{P}_3}^{3, \vec{l}_3, \lambda} \rangle d\lambda,
\end{aligned}$$

where the Fourier coefficients $\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon} := \tilde{C}_{\vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon, Q', Q''}$, the *tri-tiles* $P' := (P'_1, P'_2, P'_3)$ and $P'' := (P''_1, P''_2, P''_3)$, the *tiles* $P'_i := I_{P'_i} \times \omega_{P'_i}$ with $I_{P'_i} := I' = 2^{-k'}[n', n' + 1] =: I_{P'}$ and the frequency intervals $\omega_{P'_i} := Q'_i$ for $i = 1, 2, 3$, the *tiles* $P''_j := I_{P''_j} \times \omega_{P''_j}$ with $I_{P''_j} := I'' = 2^{-k''}[n'', n'' + 1] =: I_{P''}$ and the frequency intervals $\omega_{P''_j} := Q''_j$ for $j = 1, 2, 3$, the frequency cubes $Q_{P'} := \omega_{P'_1} \times \omega_{P'_2} \times \omega_{P'_3}$ and $Q_{P''} := \omega_{P''_1} \times \omega_{P''_2} \times \omega_{P''_3}$, \mathbb{P}' denotes a collection of such tri-tiles P' and \mathbb{P}'' denotes a collection of such tri-tiles P'' , the bi-tiles \vec{P}_1, \vec{P}_2 and \vec{P}_3 are defined by

$$\vec{P}_i := (P'_i, P''_i) = (2^{-k'}[n', n' + 1] \times Q'_i, 2^{-k''}[n'', n'' + 1] \times Q''_i)$$

for $i = 1, 2, 3$; the bi-parameter tri-tile $\vec{P} := P' \otimes P'' = (\vec{P}_1, \vec{P}_2, \vec{P}_3)$, the rectangles $I_{\vec{P}_i} := I_{P'_i} \times I_{P''_i} = I_{P'} \times I_{P''} =: I_{\vec{P}}$ for $i = 1, 2, 3$ and hence $|I_{\vec{P}}| = |I_{P'} \times I_{P''}| = |I_{\vec{P}_1}| = |I_{\vec{P}_2}| = |I_{\vec{P}_3}| = 2^{-k'} \cdot 2^{-k''}$, the double frequency cube $Q_{\vec{P}} := (Q_{P'}, Q_{P''}) = (\omega_{P'_1} \times \omega_{P'_2} \times \omega_{P'_3}, \omega_{P''_1} \times \omega_{P''_2} \times \omega_{P''_3})$, $\tilde{\mathbb{P}} := \mathbb{P}' \times \mathbb{P}''$ denotes a collection of such bi-parameter tri-tiles \vec{P} ; while the L^2 -normalized wave packets $\Phi_{P'_i}^{i, l'_i, \lambda'}$ associated with the Heisenberg boxes P'_i are defined by $\Phi_{P'_i}^{i, l'_i, \lambda'}(x_1) := \check{\phi}_{I', Q'_i, i}^{l'_i, \lambda'}(x_1) := 2^{-\frac{k'}{2}} \overline{\check{\phi}_{Q'_i, i}^{l'_i}(2^{-k'}(n' + \lambda') - x_1)}$ for $i = 1, 2, 3$, the L^2 -normalized wave packets $\Phi_{P''_i}^{i, l''_i, \lambda''}$ associated with the Heisenberg boxes P''_i are defined by $\Phi_{P''_i}^{i, l''_i, \lambda''}(x_2) := \check{\phi}_{I'', Q''_i, i}^{l''_i, \lambda''}(x_2) := 2^{-\frac{k''}{2}} \overline{\check{\phi}_{Q''_i, i}^{l''_i}(2^{-k''}(n'' + \lambda'') - x_2)}$ for $i = 1, 2, 3$, the smooth bump functions $\Phi_{\vec{P}_i}^{i, \vec{l}_i, \lambda} := \Phi_{P'_i}^{i, l'_i, \lambda'} \otimes \Phi_{P''_i}^{i, l''_i, \lambda''}$ for $i = 1, 2, 3$.

We have the following rapid decay estimates of the Fourier coefficients $\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon}$ with respect to the parameters $\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2$.

Lemma 2.10. *The Fourier coefficients $\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon}$ satisfy estimates*

$$(2.47) \quad |\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^{\varepsilon}| \lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{l}_j|)^M} \cdot \langle \log_2 \ell(Q_{P'}) \rangle^{-(1+\varepsilon)}$$

for any bi-parameter tri-tile $\vec{P} \in \tilde{\mathbb{P}}$, where M is sufficiently large.

Proof. Let $\ell(Q_{P'}) = 2^{k'}$ and $\ell(Q_{P''}) = 2^{k''}$ for $k', k'' \in \mathbb{Z}$. For any $\varepsilon > 0$, $\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2$ and $\vec{P} \in \vec{\mathbb{P}}$, we deduce from (2.42) and (2.44) that

$$(2.48) \quad \begin{aligned} \tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^\varepsilon &= \int_{\mathbb{R}^6} \tilde{m}_{Q_{P'}, Q_{P''}}^\varepsilon((2^{k'} \xi_1^1, 2^{k''} \xi_1^2), (2^{k'} \xi_2^1, 2^{k''} \xi_2^2), (2^{k'} \xi_3^1, 2^{k''} \xi_3^2)) \\ &\quad \times e^{-2\pi i(\vec{l}_1 \cdot \xi_1 + \vec{l}_2 \cdot \xi_2 + \vec{l}_3 \cdot \xi_3)} d\xi_1 d\xi_2 d\xi_3, \end{aligned}$$

where

$$(2.49) \quad \begin{aligned} \tilde{m}_{Q_{P'}, Q_{P''}}^\varepsilon((2^{k'} \xi_1^1, 2^{k''} \xi_1^2), (2^{k'} \xi_2^1, 2^{k''} \xi_2^2), (2^{k'} \xi_3^1, 2^{k''} \xi_3^2)) &:= \tilde{m}^\varepsilon(2^{k'} \bar{\xi}_1, 2^{k''} \bar{\xi}_2) \\ &\quad \times \phi_{\omega_{P'_1} \times \omega_{P'_2}}(2^{k'} \bar{\xi}_1) \tilde{\phi}_{\omega_{P'_3}, 3}(2^{k'} \bar{\xi}_1) \phi_{\omega_{P''_1} \times \omega_{P''_2}}(2^{k''} \bar{\xi}_2) \tilde{\phi}_{\omega_{P''_3}, 3}(2^{k''} \bar{\xi}_2). \end{aligned}$$

Observe that $\text{supp}(\phi_{\omega_{P'_1} \times \omega_{P'_2}}(\bar{\xi}_1) \tilde{\phi}_{\omega_{P'_3}, 3}(\xi_3^1) \phi_{\omega_{P''_1} \times \omega_{P''_2}}(\bar{\xi}_2) \tilde{\phi}_{\omega_{P''_3}, 3}(\xi_3^2)) \subseteq Q_{P'} \times Q_{P''}$, we have that $\text{supp}(\phi_{\omega_{P'_1} \times \omega_{P'_2}}(2^{k'} \bar{\xi}_1) \tilde{\phi}_{\omega_{P'_3}, 3}(2^{k'} \bar{\xi}_1) \phi_{\omega_{P''_1} \times \omega_{P''_2}}(2^{k''} \bar{\xi}_2) \tilde{\phi}_{\omega_{P''_3}, 3}(2^{k''} \bar{\xi}_2)) \subseteq Q_{P'}^0 \times Q_{P''}^0$, where cubes $Q_{P'}^0$ and $Q_{P''}^0$ are defined by

$$(2.50) \quad Q_{P'}^0 = \omega_{P'_1}^0 \times \omega_{P'_2}^0 \times \omega_{P'_3}^0 := \{(\xi_1^1, \xi_2^1, \xi_3^1) \in \mathbb{R}^3 : (2^{k'} \xi_1^1, 2^{k'} \xi_2^1, 2^{k'} \xi_3^1) \in Q_{P'}\},$$

$$(2.51) \quad Q_{P''}^0 = \omega_{P''_1}^0 \times \omega_{P''_2}^0 \times \omega_{P''_3}^0 := \{(\xi_1^2, \xi_2^2, \xi_3^2) \in \mathbb{R}^3 : (2^{k''} \xi_1^2, 2^{k''} \xi_2^2, 2^{k''} \xi_3^2) \in Q_{P''}\}$$

and satisfy $|Q_{P'}^0| \simeq |Q_{P''}^0| \simeq 1$. From the properties of the *Whitney squares* we constructed above, one obtains that $\text{dist}(2^{k'} \bar{\xi}_1, \Gamma_1) \simeq 2^{k'}$ for any $\bar{\xi}_1 \in \omega_{P'_1}^0 \times \omega_{P'_2}^0$ and $\text{dist}(2^{k''} \bar{\xi}_2, \Gamma_2) \simeq 2^{k''}$ for any $\bar{\xi}_2 \in \omega_{P''_1}^0 \times \omega_{P''_2}^0$.

By taking advantage of the estimates (1.12) for symbol $\tilde{m}^\varepsilon(\bar{\xi})$, one can deduce from (2.48), (2.49) and integrating by parts sufficiently many times that

$$\begin{aligned} |\tilde{C}_{Q_{\vec{P}}, \vec{l}_1, \vec{l}_2, \vec{l}_3}^\varepsilon| &\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{l}_j|)^M} \\ &\quad \times \int_{Q_{P'}^0 \times Q_{P''}^0} |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} [\tilde{m}_{Q_{P'}, Q_{P''}}^\varepsilon((2^{k'} \xi_1^1, 2^{k''} \xi_1^2), (2^{k'} \xi_2^1, 2^{k''} \xi_2^2), (2^{k'} \xi_3^1, 2^{k''} \xi_3^2))]| d\xi_1 d\xi_2 d\xi_3 \\ &\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{l}_j|)^M} \int_{\omega_{P'_1}^0 \times \omega_{P'_2}^0} \text{dist}(2^{k''} \bar{\xi}_2, \Gamma_2)^{|\alpha''|} \\ &\quad \times \int_{\omega_{P'_1}^0 \times \omega_{P'_2}^0} \text{dist}(2^{k'} \bar{\xi}_1, \Gamma_1)^{|\alpha'|} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} \tilde{m}^\varepsilon(2^{k'} \bar{\xi}_1, 2^{k''} \bar{\xi}_2)| d\bar{\xi}_1 d\bar{\xi}_2 \\ &\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{l}_j|)^M} \cdot 2^{-2k'} 2^{-2k''} \int_{\omega_{P'_1}^0 \times \omega_{P'_2}^0} \int_{\omega_{P''_1}^0 \times \omega_{P''_2}^0} \\ &\quad \text{dist}(\bar{\xi}_2, \Gamma_2)^{|\alpha''|} \cdot \text{dist}(\bar{\xi}_1, \Gamma_1)^{|\alpha'|} |\partial_{\xi_1}^{\alpha'} \partial_{\xi_2}^{\alpha''} \tilde{m}^\varepsilon(\bar{\xi}_1, \bar{\xi}_2)| d\bar{\xi}_1 d\bar{\xi}_2 \\ &\lesssim \prod_{j=1}^3 \frac{1}{(1 + |\vec{l}_j|)^M} \cdot \langle \log_2 \ell(Q_{P'}) \rangle^{-(1+\varepsilon)}, \end{aligned}$$

where the multi-indices $\alpha_i := (\alpha_i^1, \alpha_i^2)$ for $i = 1, 2, 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3| = M$ are sufficiently large, the multi-indices $\alpha' := (\alpha'_1, \alpha'_2, \alpha'_3)$, $\alpha'' := (\alpha''_1, \alpha''_2, \alpha''_3)$ with $\alpha'_i \leq \alpha_i^1$ and $\alpha''_j \leq \alpha_j^2$ for $i, j = 1, 2, 3$. This ends our proof of estimates (2.47). \square

Observe that the rapid decay with respect to the parameters $\vec{l}_1, \vec{l}_2, \vec{l}_3 \in \mathbb{Z}^2$ in (2.47) is acceptable for summation, all the functions $\Phi_{P'_i}^{i, l'_i, \lambda'}$ ($i = 1, 2, 3$) are L^2 normalized and are wave packets associated with the Heisenberg boxes P'_i uniformly with respect to the parameters l'_i and all the functions $\Phi_{P''_j}^{j, l''_j, \lambda''}$ ($j = 1, 2, 3$) are L^2 normalized and are wave packets associated with the Heisenberg boxes P''_j uniformly with respect to the parameters l''_j , therefore we only need to consider from now on the part of the trilinear form $\dot{\Lambda}_{\tilde{m}_{\mathbb{I}, \mathbb{I}}}^{(2)}(f_1, f_2, f_3)$ defined in (2.46) corresponding to $\vec{l}_1 = \vec{l}_2 = \vec{l}_3 = \vec{0}$:

$$(2.52) \quad \dot{\Lambda}_{\tilde{m}_{\mathbb{I}, \mathbb{I}}}^{(2)}(f_1, f_2, f_3) := \int_0^1 \int_0^1 \sum_{\vec{P} \in \tilde{\mathbb{P}}} \frac{\tilde{C}_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \lambda} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \lambda} \rangle \langle f_3, \Phi_{\vec{P}_3}^{3, \lambda} \rangle d\lambda,$$

where $\tilde{C}_{Q_{\vec{P}}}^\varepsilon := \tilde{C}_{Q_{\vec{P}}, \vec{0}, \vec{0}, \vec{0}}^\varepsilon$, parameters $\lambda = (\lambda', \lambda'')$ and $\Phi_{\vec{P}_i}^{i, \lambda} := \Phi_{\vec{P}_i}^{i, \vec{0}, \lambda}$ for $i = 1, 2, 3$.

The tri-tiles $P' = (P'_1, P'_2, P'_3)$ in collection \mathbb{P}' also satisfy the same properties (as $P'' \in \mathbb{P}''$) described in Remark 2.5. It is not difficult to observe that both the collections of tri-tiles \mathbb{P}' and \mathbb{P}'' can be written as a finite union of sparse collections of rank 1, thus we may assume further that \mathbb{P}' and \mathbb{P}'' are sparse collection of rank 1 from now on.

The bilinear operator corresponding to the trilinear form $\dot{\Lambda}_{\tilde{m}_{\mathbb{I}, \mathbb{I}}}^{(2)}(f_1, f_2, f_3)$ can be written as

$$(2.53) \quad \dot{\tilde{\Pi}}_{\tilde{\mathbb{P}}}^\varepsilon(f_1, f_2)(x) = \int_0^1 \int_0^1 \sum_{\vec{P} \in \tilde{\mathbb{P}}} \frac{\tilde{C}_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^{1, \lambda} \rangle \langle f_2, \Phi_{\vec{P}_2}^{2, \lambda} \rangle \Phi_{\vec{P}_3}^{3, \lambda}(x) d\lambda.$$

Since $\dot{\tilde{\Pi}}_{\tilde{\mathbb{P}}}^\varepsilon(f_1, f_2)$ is an average of some discrete bilinear model operators depending on the parameters $\lambda = (\lambda_1, \lambda_2) \in [0, 1]^2$, it is enough to prove the Hölder-type L^p estimates for each of them, uniformly with respect to parameters $\lambda = (\lambda_1, \lambda_2)$. From now on, we will do this in the particular case when the parameters $\lambda = (\lambda_1, \lambda_2) = (0, 0)$, but the same argument works in general. By Fatou's lemma, we can also restrict the summation in the definition (2.53) of $\dot{\tilde{\Pi}}_{\tilde{\mathbb{P}}}^\varepsilon(f_1, f_2)$ on collection $\tilde{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ with arbitrary finite collections \mathbb{P}' and \mathbb{P}'' of tri-tiles, and prove the estimates are uniform with respect to different choices of the set $\tilde{\mathbb{P}}$.

Definition 2.11. A finite collection $\tilde{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ of bi-parameter tri-tiles is said to be sparse and rank 1, if both the finite collections \mathbb{P}' and \mathbb{P}'' are sparse and rank 1.

Therefore, one can reduce the bilinear operator $\dot{\tilde{\Pi}}_{\tilde{\mathbb{P}}}^\varepsilon$ further to the discrete bilinear model operator $\tilde{\Pi}_{\tilde{\mathbb{P}}}^\varepsilon$ defined by

$$(2.54) \quad \tilde{\Pi}_{\tilde{\mathbb{P}}}^\varepsilon(f_1, f_2)(x) := \sum_{\vec{P} \in \tilde{\mathbb{P}}} \frac{\tilde{C}_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^1 \rangle \langle f_2, \Phi_{\vec{P}_2}^2 \rangle \Phi_{\vec{P}_3}^3(x),$$

where $\Phi_{\vec{P}_j}^j := \Phi_{\vec{P}_j}^{j,(0,0)}$ for $j = 1, 2, 3$ respectively, the finite set $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$ is an arbitrary sparse collection (of bi-parameter tri-tiles) of rank 1. As have discussed above, we now reach a conclusion that the proof of Theorem 1.4 can be reduced to proving the following L^p estimates for discrete bilinear model operators $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$.

Proposition 2.12. *If the finite set $\vec{\mathbb{P}}$ is an arbitrary sparse collection of rank 1, then operator $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ given by (2.54) maps $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ boundedly for any $1 < p_1, p_2 \leq \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{2}{3} < p < \infty$. Moreover, the implicit constants in the bounds depend only on ε, p_1, p_2, p and are independent of the particular finite sparse collection $\vec{\mathbb{P}}$ of rank 1.*

2.2. Multi-linear interpolations. First, let's review the following terminologies and definitions of multi-linear interpolation arguments from [25, 26].

Definition 2.13. ([25, 26]) An n -tuple $\beta = (\beta_1, \dots, \beta_n)$ is said to be *admissible* if and only if $\beta_j < 1$ for every $1 \leq j \leq n$, $\sum_{j=1}^n \beta_j = 1$ and there is at most one index j for which $\beta_j < 0$. An index j is called *good* if $\beta_j \geq 0$ and *bad* if $\beta_j < 0$. A *good tuple* is an admissible tuple that contains only good indices; a *bad tuple* is an admissible tuple that contains precisely one bad index.

Definition 2.14. ([26]) Let E, E' be sets of finite measure. We say that E' is a *major subset* of E if $E' \subseteq E$ and $|E'| \geq \frac{1}{2}|E|$.

Definition 2.15. ([25, 26]) If $\beta = (\beta_1, \dots, \beta_n)$ is an admissible tuple, we say that an n -linear form Λ is of *restricted weak type β* if and only if, for every sequence E_1, \dots, E_n of measurable sets with positive and finite measure, there exists a major subset E'_j of E_j for each bad index j (one or none) such that

$$(2.55) \quad |\Lambda(f_1, \dots, f_n)| \lesssim |E_1|^{\beta_1} \dots |E_n|^{\beta_n}$$

for every measurable functions $|f_i| \leq \chi_{E'_i}$ ($i = 1, \dots, n$), where we adopt the convention $E'_i = E_i$ for good indices i . If β is bad with bad index j_0 , and it happens that one can choose the major subset $E'_{j_0} \subseteq E_{j_0}$ in a way that depends only on the measurable sets E_1, \dots, E_n and not on β , we say that Λ is of *uniformly restricted weak type*.

Definition 2.16. ([25]) Let $1 < p_1, p_2 \leq \infty$ and $0 < p < \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. An arbitrary bilinear operator T is said to be of the restricted weak type (p_1, p_2, p) if and only if for all measurable sets E_1, E_2, E of finite measure there exists $E' \subseteq E$ with $|E'| \simeq |E|$ such that

$$(2.56) \quad \left| \int_{\mathbb{R}^d} T(f_1, f_2)(x) f(x) dx \right| \lesssim |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}} |E'|^{\frac{1}{p}}$$

for every $|f_1| \leq \chi_{E_1}, |f_2| \leq \chi_{E_2}$ and $|f| \leq \chi_{E'}$.

By using multi-linear interpolation (see [13, 15, 25, 26]) and the symmetry of operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ and $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$, we can reduce further the proof of Proposition 2.9 and Proposition 2.12 to proving the following restricted weak type estimates for the model operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ and $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$.

Proposition 2.17. *Let p_1 and p_2 be such that p_1 is strictly larger than 1 and arbitrarily close to 1 and p_2 is strictly smaller than 2 and arbitrarily close to 2 and such that for $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$, one has $\frac{2}{3} < p < 1$. Then both the model operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ and $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ defined in (2.33) and (2.54) are of the restricted weak type (p_1, p_2, p) . Moreover, the implicit constants in the bounds depend only on ε, p_1, p_2, p and are independent of the particular choice of the finite collection $\vec{\mathbb{P}}$.*

Indeed, first we should observe that if p_1, p_2, p are as in Proposition 2.9 and 2.12 then the 3-tuple $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in the interior of the convex hull of the following six extremal points: $\beta^1 := (-\frac{1}{2}, \frac{1}{2}, 1)$, $\beta^2 := (-\frac{1}{2}, 1, \frac{1}{2})$, $\beta^3 := (\frac{1}{2}, -\frac{1}{2}, 1)$, $\beta^4 := (1, -\frac{1}{2}, \frac{1}{2})$, $\beta^5 := (\frac{1}{2}, 1, -\frac{1}{2})$ and $\beta^6 := (1, \frac{1}{2}, -\frac{1}{2})$. Then, if we assume that Proposition 2.17 has been proved, from the symmetry of operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ and $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ and their adjoints, we deduce that both the tri-linear forms associated to bilinear operators $\Pi_{\vec{\mathbb{P}}}^\varepsilon$ and $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ are of *uniformly restricted weak type* β for 3-tuples $\beta = (\beta_1, \beta_2, \beta_3)$ arbitrarily close to the six extremal points β^1, \dots, β^6 inside the convex hull of them and satisfying if β_j is close to $\frac{1}{2}$ for some $j = 1, 2, 3$ then β_j is strictly larger than $\frac{1}{2}$. By using multi-linear interpolation lemma 9.4 and 9.6 in [25] or lemma 3.8 in [26], we first obtain restricted weak type estimates of Λ for good tuples inside the smaller convex hull of the three coordinate points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. After that, we use the interpolation lemma 9.5 in [25] or lemma 3.10 in [26] to obtain restricted weak type estimates of Λ for bad tuples and finally conclude that restricted weak type estimates of Λ hold for all tuples β inside the convex hull of the six extremal points β^1, \dots, β^6 .

It only remains to convert these restricted weak type estimates into strong type estimates. To do this, one just has to apply (exactly as in [26]) the multi-linear Marcinkiewicz interpolation theorem in [15] in the case of good tuples and the interpolation lemma 3.11 in [26] in the case of bad tuples. This ends the proof of Proposition 2.9 and 2.12, and as a consequence, completes the proof of our main results, Theorem 1.3 and 1.4. Therefore, we only have the task of proving Proposition 2.17 from now on.

3. TREES, L^2 SIZES AND L^2 ENERGIES

3.1. Trees. We should recall that for discrete bilinear paraproducts, the frequency intervals have been already organized with the lacunary properties (see [25, 27, 29]), we could use square function and Maximal function estimates to handle the corresponding terms easily, at least in the Banach case. By the properties of the collection \mathbb{P}'' of tri-tiles we have explained in Remark 2.5, we can organize our collections of tri-tiles $\mathbb{P}', \mathbb{P}''$ into trees as in [9], which satisfy lacunary properties about certain frequency. We review the following standard definitions and properties for trees from [28].

Definition 3.1. Let \mathbb{P} be a sparse rank-1 collection of tri-tiles and $j \in \{1, 2, 3\}$. A sub-collection $T \subseteq \mathbb{P}$ is called a *j-tree* if and only if there exists a tri-tile P_T (called the top of the tree) such that

$$(3.1) \quad P_j \leq P_{T,j}$$

for every $P \in T$.

Remark 3.2. Note that a tree does not necessarily have to contain the corresponding top P_T . From now on, we will write I_T and $\omega_{T,j}$ for I_{P_T} and $\omega_{P_T,j}$ for $j = 1, 2, 3$ respectively. Then, we simply say that T is a tree if it is a j -tree for some $j = 1, 2, 3$.

For every given dyadic interval I_0 , there are potentially many tri-tiles P in collections \mathbb{P}' and \mathbb{P}'' with the property that $I_P = I_0$. Due to this *extra degree of freedom* in frequency, we have infinitely many trees in our collections \mathbb{P}' and \mathbb{P}'' . We need to estimate each of these trees separately, and then add all these estimates together, by using the *almost orthogonality* conditions for distinct trees. This motivates the following definition.

Definition 3.3. Let $1 \leq i \leq 3$. A finite sequence of trees T_1, \dots, T_M is said to be a *chain of strongly i -disjoint trees* if and only if

- (i) $P_i \neq P'_i$ for every $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 \neq l_2$;
- (ii) whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 \neq l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$, then if $|\omega_{P_i}| < |\omega_{P'_i}|$ one has $I_{P'} \cap I_{T_{l_1}} = \emptyset$ and if $|\omega_{P'_i}| < |\omega_{P_i}|$ one has $I_P \cap I_{T_{l_2}} = \emptyset$;
- (iii) whenever $P \in T_{l_1}$ and $P' \in T_{l_2}$ with $l_1 < l_2$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$, then if $|\omega_{P_i}| = |\omega_{P'_i}|$ one has $I_{P'} \cap I_{T_{l_1}} = \emptyset$.

3.2. L^2 sizes and L^2 energies. Following [28], we give the definitions of standard norms on sequences of tiles as follows.

Definition 3.4. Let \mathbb{P} be a finite collection of tri-tiles, $j \in \{1, 2, 3\}$, and f be an arbitrary function. We define the *size* of the sequence $(\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}$ by

$$(3.2) \quad \text{size}_j((\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}) := \sup_{T \subseteq \mathbb{P}} \left(\frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi_{P_j}^j \rangle|^2 \right)^{\frac{1}{2}},$$

where T ranges over all trees in \mathbb{P} that are i -trees for some $i \neq j$. For $j = 1, 2, 3$, we define the *energy* of the sequence $(\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}$ by

$$(3.3) \quad \text{energy}_j((\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}) := \sup_{n \in \mathbb{Z}} \sup_{\mathbb{T}} 2^n \left(\sum_{T \in \mathbb{T}} |I_T| \right)^{\frac{1}{2}},$$

where now \mathbb{T} ranges over all chains of strongly j -disjoint trees in \mathbb{P} (which are i -trees for some $i \neq j$) having the property that

$$(3.4) \quad \left(\sum_{P \in T} |\langle f, \Phi_{P_j}^j \rangle|^2 \right)^{\frac{1}{2}} \geq 2^n |I_T|^{\frac{1}{2}}$$

for all $T \in \mathbb{T}$ and such that

$$(3.5) \quad \left(\sum_{P \in T'} |\langle f, \Phi_{P_j}^j \rangle|^2 \right)^{\frac{1}{2}} \leq 2^{n+1} |I_{T'}|^{\frac{1}{2}}$$

for all subtrees $T' \subseteq T \in \mathbb{T}$.

The *size* measures the extent to which the sequences $(\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}$ ($j = 1, 2, 3$) can concentrate on a single tree and should be thought of as a phase-space variant of the *BMO* norm. The *energy* is a phase-space variant of the L^2 norm. As the notation suggests, the number $\langle f, \Phi_{P_j}^j \rangle$ should be thought of as being associated with the tile P_j ($j = 1, 2, 3$) rather than the full tri-tile P .

Let \mathbb{P} be a finite collection of tri-tiles. Denote by $\Pi_{\mathbb{P}}$ the discrete bilinear operator given by

$$\Pi_{\mathbb{P}}(f_1, f_2)(x) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{\frac{1}{2}}} \langle f_1, \Phi_{P_1}^1 \rangle \langle f_2, \Phi_{P_2}^2 \rangle \Phi_{P_3}^3(x).$$

The following proposition provides a way of estimating the trilinear form associated with bilinear operator $\Pi_{\mathbb{P}}(f_1, f_2)$. We define

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) := \int_{\mathbb{R}} T_{\mathbb{P}}(f_1, f_2)(x) f_3(x) dx.$$

Proposition 3.5. ([28]) *Let \mathbb{P} be a finite collection of tri-tiles. Then*

$$(3.6) \quad |\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \lesssim \prod_{j=1}^3 (\text{size}_j((\langle f_j, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}))^{\theta_j} (\text{energy}_j((\langle f_j, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}))^{1-\theta_j}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$; the implicit constants depend on the θ_j but are independent of the other parameters.

3.3. Estimates for sizes and energies. In order to apply Proposition 3.5, we need to estimate further the sizes and energies appearing on the right-hand side of (3.6).

Lemma 3.6. ([25, 28]) *Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then one has*

$$(3.7) \quad \text{size}_j((\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}) \lesssim \sup_{P \in \mathbb{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f| \tilde{\chi}_{I_P}^M dx$$

for every $M > 0$, where the approximate cutoff function $\tilde{\chi}_{I_P}^M(x) := (1 + \frac{\text{dist}(x, I_P)}{|I_P|})^{-M}$ and the implicit constants depend on M .

Lemma 3.7. (Bessel-type estimates, [28]). *Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then*

$$(3.8) \quad \text{energy}_j((\langle f, \Phi_{P_j}^j \rangle)_{P \in \mathbb{P}}) \lesssim \|f\|_{L^2}.$$

4. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3 by carrying out the proof of Proposition 2.17 for model operators $\Pi_{\vec{\mathbb{P}}}^{\varepsilon}$ defined in (2.33) with $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$.

Fix indices p_1, p_2, p as in the hypothesis of Proposition 2.17. Fix arbitrary measurable sets E_1, E_2, E_3 of finite measure (by using the scaling invariance of $\Pi_{\vec{\mathbb{P}}}^{\varepsilon}$, we can assume further that $|E_3| = 1$). Our goal is to find $E'_3 \subseteq E_3$ with $|E'_3| \simeq |E_3| = 1$ such that, for any functions $|f_1| \leq \chi_{E_1}$, $|f_2| \leq \chi_{E_2}$ and $|f_3| \leq \chi_{E'_3}$, one has the corresponding trilinear forms $\Lambda_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2, f_3)$ defined by

$$(4.1) \quad \Lambda_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2, f_3) := \int_{\mathbb{R}^2} \Pi_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2)(x) f_3(x) dx$$

satisfy estimates

$$(4.2) \quad |\Lambda_{\vec{\mathbb{P}}}^{\varepsilon}(f_1, f_2, f_3)| = \left| \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{C_{Q_{\vec{P}}}^{\varepsilon}}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^1 \rangle \langle f_2, \Phi_{\vec{P}_2}^2 \rangle \langle f_3, \Phi_{\vec{P}_3}^3 \rangle \right| \lesssim_{\varepsilon, p, p_1, p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}},$$

where p_1 is larger than but close to 1, while p_2 is smaller than but close to 2.

From [29], we can find the following generic decomposition lemma.

Lemma 4.1. *Let $J \subseteq \mathbb{R}$ be a fixed interval. Then every smooth bump function ϕ_J adapted to J can be naturally decomposed as follows:*

$$\phi_J = \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \phi_J^\ell,$$

where for every $\ell \in \mathbb{N}$, ϕ_J^ℓ is also a bump function adapted to J but having the additional property that $\text{supp}(\phi_J^\ell) \subseteq 2^\ell J$. If in addition we assume that $\int_{\mathbb{R}} \phi_J(x) dx = 0$, then the functions ϕ_J^ℓ can be chosen such that $\int_{\mathbb{R}} \phi_J^\ell(x) dx = 0$ for every $\ell \in \mathbb{N}$.

We use $2^\ell J$ to denote the interval having the same center as J but with length 2^ℓ times that of J hereafter.

By using Lemma 4.1, we can estimate the left-hand side of (4.2) by

$$(4.3) \quad |\Lambda_{\mathbb{P}}^\varepsilon(f_1, f_2, f_3)| \lesssim \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \Lambda_{\mathbb{P}}^{\varepsilon, \ell}(f_1, f_2, f_3).$$

The tri-linear forms $\Lambda_{\mathbb{P}}^{\varepsilon, \ell}(f_1, f_2, f_3)$ ($\ell \in \mathbb{N}$) are defined by

$$(4.4) \quad \Lambda_{\mathbb{P}}^{\varepsilon, \ell}(f_1, f_2, f_3) := \sum_{\bar{P} \in \mathbb{P}} \frac{|C_{Q_{\bar{P}}}^\varepsilon|}{|I_{\bar{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi_{\bar{P}_1}^1 \rangle| |\langle f_2, \Phi_{\bar{P}_2}^2 \rangle| |\langle f_3, \Phi_{\bar{P}_3}^{3, \ell} \rangle|,$$

where the new bi-parameter wave packets $\Phi_{\bar{P}_3}^{3, \ell} := \Phi_{\bar{P}_3'}^{3, \ell} \otimes \Phi_{\bar{P}_3''}^3$ with additional property that $\text{supp}(\Phi_{\bar{P}_3'}^{3, \ell}) \subseteq 2^\ell I_{\bar{P}_3'} = 2^\ell I_{\bar{P}_3'}$.

For every $\ell \in \mathbb{N}$, we define the sets as follows:

$$(4.5) \quad \Omega_{-10\ell} := \{x \in \mathbb{R}^2 : MM(\frac{\chi_{E_1}}{|E_1|})(x) > C2^{10\ell}\} \cup \{x \in \mathbb{R}^2 : MM(\frac{\chi_{E_2}}{|E_2|})(x) > C2^{10\ell}\},$$

and

$$(4.6) \quad \tilde{\Omega}_{-10\ell} := \{x \in \mathbb{R}^2 : MM(\chi_{\Omega_{-10\ell}})(x) > 2^{-\ell}\},$$

where the double maximal operator MM is given by

$$(4.7) \quad MM(h)(x, y) := \sup_{\substack{\text{dyadic rectangle } R \\ (x, y) \in R}} \frac{1}{|R|} \int_R |h(u, v)| du dv.$$

Finally, we define the exceptional set

$$(4.8) \quad U := \bigcup_{\ell \in \mathbb{N}} \tilde{\Omega}_{-10\ell}.$$

It is clear that $|U| < \frac{1}{10}$ if C is a large enough constant, which we fix from now on. Then, we define $E'_3 := E_3 \setminus U$ and observe that $|E'_3| \simeq 1$.

Now fix $\ell \in \mathbb{N}$, and split the trilinear form $\Lambda_{\mathbb{P}}^{\varepsilon, \ell}(f_1, f_2, f_3)$ defined in (4.4) into two parts as follows:

$$\begin{aligned}
 \Lambda_{\mathbb{P}}^{\varepsilon, \ell}(f_1, f_2, f_3) &= \sum_{\substack{\vec{P} \in \mathbb{P}: \\ I_{\vec{P}} \cap \Omega_{-10\ell}^c \neq \emptyset}} \frac{|C_{Q_{\vec{P}}}^{\varepsilon}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi_{\vec{P}_1}^1 \rangle| |\langle f_2, \Phi_{\vec{P}_2}^2 \rangle| |\langle f_3, \Phi_{\vec{P}_3}^{3, \ell} \rangle| \\
 &+ \sum_{\substack{\vec{P} \in \mathbb{P}: \\ I_{\vec{P}} \cap \Omega_{-10\ell}^c = \emptyset}} \frac{|C_{Q_{\vec{P}}}^{\varepsilon}|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi_{\vec{P}_1}^1 \rangle| |\langle f_2, \Phi_{\vec{P}_2}^2 \rangle| |\langle f_3, \Phi_{\vec{P}_3}^{3, \ell} \rangle| \\
 &=: \Lambda_{\mathbb{P}, I}^{\varepsilon, \ell}(f_1, f_2, f_3) + \Lambda_{\mathbb{P}, II}^{\varepsilon, \ell}(f_1, f_2, f_3),
 \end{aligned}
 \tag{4.9}$$

where the notation A^c denotes the complementary set of a set A .

4.1. Estimates for trilinear form $\Lambda_{\mathbb{P}, I}^{\varepsilon, \ell}(f_1, f_2, f_3)$. We can decompose the collection $\widetilde{\mathbb{P}}$ of tri-tiles into

$$\widetilde{\mathbb{P}}' = \bigcup_{k' \in \mathbb{Z}} \widetilde{\mathbb{P}}'_{k'},
 \tag{4.10}$$

where

$$\widetilde{\mathbb{P}}'_{k'} := \{\widetilde{P}' \in \widetilde{\mathbb{P}}' : |I_{\widetilde{P}'}| = 2^{-k'}\}.
 \tag{4.11}$$

As a consequence, we can split the trilinear form $\Lambda_{\mathbb{P}, I}^{\varepsilon, \ell}(f_1, f_2, f_3)$ into

$$\begin{aligned}
 \Lambda_{\mathbb{P}, I}^{\varepsilon, \ell}(f_1, f_2, f_3) &= \sum_{k' \in \mathbb{Z}} \sum_{\substack{\vec{P} \in \widetilde{\mathbb{P}}'_{k'} \times \mathbb{P}'': \\ I_{\vec{P}} \cap \Omega_{-10\ell}^c \neq \emptyset}} |C_{Q_{\vec{P}}}^{\varepsilon}| \frac{|I_{\widetilde{P}'}|}{|I_{P''}|^{\frac{1}{2}}} |\langle \frac{f_1, \Phi_{\widetilde{P}_1}^1}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_1}^1 \rangle| \\
 &\quad \times |\langle \frac{f_2, \Phi_{\widetilde{P}_2}^2}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_2}^2 \rangle| |\langle \frac{f_3, \Phi_{\widetilde{P}_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3}^3 \rangle|.
 \end{aligned}
 \tag{4.12}$$

Note that by Lemma 2.4, we can estimate the Fourier coefficients $C_{Q_{\vec{P}}}^{\varepsilon} := C_{Q_{\vec{P}}, \vec{0}, \vec{0}, \vec{0}}^{\varepsilon}$ for each $\vec{P} \in \widetilde{\mathbb{P}}'_{k'} \times \mathbb{P}''$ ($k' \in \mathbb{Z}$) by

$$|C_{Q_{\vec{P}}}^{\varepsilon}| \lesssim C_{k'}^{\varepsilon} \quad \text{with} \quad \sum_{k' \in \mathbb{Z}} C_{k'}^{\varepsilon} \lesssim_{\varepsilon} 1.
 \tag{4.13}$$

For each fixed $\widetilde{P}' \in \widetilde{\mathbb{P}}'$, we define the sub-collection

$$\mathbb{P}''_{\widetilde{P}'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \cap \Omega_{-10\ell}^c \neq \emptyset\}.$$

Therefore, by using Proposition 3.5, we derive the following estimates

$$\begin{aligned}
\Lambda_{\mathbb{P},I}^{\varepsilon,\ell}(f_1, f_2, f_3) &\lesssim \sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon \sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} |I_{\widetilde{P}'}| \left[\prod_{j=1}^2 (energy_j(\langle \frac{\langle f_j, \Phi_{\widetilde{P}'}^j \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_j''}^j \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \right)^{1-\theta_j} \\
(4.14) \quad &\times (size_j(\langle \frac{\langle f_j, \Phi_{\widetilde{P}'}^j \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_j''}^j \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}})^{\theta_j} (size_3(\langle \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3''}^3 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}})^{\theta_3} \\
&\times (energy_3(\langle \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3''}^3 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}})^{1-\theta_3}
\end{aligned}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the right-hand side of (4.14), note that $I_{\widetilde{P}} \cap \Omega_{-10\ell}^c \neq \emptyset$ and $supp f_3 \subseteq E'_3 \subseteq \mathbb{R}^2 \setminus U$, we apply the *size* estimates in Lemma 3.6 and get for each $\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}$,

$$(4.15) \quad size_1(\langle \frac{\langle f_1, \Phi_{\widetilde{P}'}^1 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_1''}^1 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \sup_{P'' \in \mathbb{P}_{\widetilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_1, \Phi_{\widetilde{P}'}^1 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M dx \lesssim 2^{10\ell} |E_1|,$$

$$(4.16) \quad size_2(\langle \frac{\langle f_2, \Phi_{\widetilde{P}'}^2 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_2''}^2 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \sup_{P'' \in \mathbb{P}_{\widetilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_2, \Phi_{\widetilde{P}'}^2 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M dx \lesssim 2^{10\ell} |E_2|,$$

$$(4.17) \quad size_3(\langle \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3''}^3 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \sup_{P'' \in \mathbb{P}_{\widetilde{P}'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M dx \lesssim 1,$$

where $M > 0$ is sufficiently large. By applying the *energy* estimates in Lemma 3.7 and Hölder estimates, we have for each $\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}$,

$$(4.18) \quad energy_1(\langle \frac{\langle f_1, \Phi_{\widetilde{P}'}^1 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_1''}^1 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \left\| \frac{\langle f_1, \Phi_{\widetilde{P}'}^1 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E_1} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2 \right)^{\frac{1}{2}},$$

$$(4.19) \quad energy_2(\langle \frac{\langle f_2, \Phi_{\widetilde{P}'}^2 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_2''}^2 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \left\| \frac{\langle f_2, \Phi_{\widetilde{P}'}^2 \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E_2} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2 \right)^{\frac{1}{2}},$$

$$(4.20) \quad energy_3(\langle \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3''}^3 \rangle)_{P'' \in \mathbb{P}_{\widetilde{P}'}} \lesssim \left\| \frac{\langle f_3, \Phi_{\widetilde{P}'}^{3,\ell} \rangle}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} \lesssim \left(\int_{E'_3} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100,\ell}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2 \right)^{\frac{1}{2}},$$

where the approximate cutoff function $\widetilde{\chi}_{I_{\widetilde{P}'}}^{100,\ell}(x_1)$ decays rapidly (of order 100) away from the interval $I_{\widetilde{P}'}$ at scale $|I_{\widetilde{P}'}|$ and satisfies additional property that $supp \widetilde{\chi}_{I_{\widetilde{P}'}}^{100,\ell} \subseteq 2^\ell I_{\widetilde{P}'}$.

Now we insert the size and energy estimates (4.15)-(4.20) into (4.14) and get

$$(4.21) \quad \Lambda_{\mathbb{P},I}^{\varepsilon,\ell}(f_1, f_2, f_3) \lesssim 2^{10\ell} |E_1|^{\theta_1} |E_2|^{\theta_2} \sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon \sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \left(\int_{E_1} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_1}{2}} \left(\int_{E_2} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_2}{2}} \left(\int_{E_3'} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100,\ell} dx \right)^{\frac{1-\theta_3}{2}}.$$

Since $|I_{\widetilde{P}'}| = 2^{-k'}$ for every $\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}$, all the dyadic intervals $I_{\widetilde{P}'}$ are disjoint, thus by using Hölder inequality, we can estimate the inner sum in the right-hand side of (4.21) by

$$(4.22) \quad \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_1} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_1}{2}} \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_2} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_2}{2}} \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_3'} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100,\ell} dx \right)^{\frac{1-\theta_3}{2}} \lesssim |E_1|^{\frac{1-\theta_1}{2}} |E_2|^{\frac{1-\theta_2}{2}}.$$

Combining the estimates (4.13), (4.21) and (4.22), we arrive at

$$(4.23) \quad \Lambda_{\mathbb{P},I}^{\varepsilon,\ell}(f_1, f_2, f_3) \lesssim 2^{10\ell} |E_1|^{\theta_1} |E_2|^{\theta_2} |E_1|^{\frac{1-\theta_1}{2}} |E_2|^{\frac{1-\theta_2}{2}} \sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon \lesssim_{\varepsilon, \theta_1, \theta_2, \theta_3} 2^{20\ell} |E_1|^{\frac{1+\theta_1}{2}} |E_2|^{\frac{1+\theta_2}{2}}$$

for every $\ell \in \mathbb{N}$ and $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponents $\frac{2}{1+\theta_1} = p_1$ strictly larger than 1 and close to 1 and $\frac{2}{1+\theta_2} = p_2$ strictly smaller than 2 and close to 2. We finally get the estimate

$$(4.24) \quad \Lambda_{\mathbb{P},I}^{\varepsilon,\ell}(f_1, f_2, f_3) \lesssim_{\varepsilon, p, p_1, p_2} 2^{10\ell} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for every $\ell \in \mathbb{N}$, $\varepsilon > 0$ and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

4.2. Estimates for trilinear form $\Lambda_{\mathbb{P},II}^{\varepsilon,\ell}(f_1, f_2, f_3)$. One can observe that if $I_{\widetilde{P}} \subseteq \Omega_{-10\ell}$, then $2^\ell I_{\widetilde{P}'} \times I_{P''} \subseteq \widetilde{\Omega}_{-10\ell}$. Therefore, for each fixed $\widetilde{P}' \in \widetilde{\mathbb{P}}$, we define the corresponding sub-collection of \mathbb{P}'' by

$$\mathbb{P}_{\widetilde{P}'}'' := \{P'' \in \mathbb{P}'' : I_{\widetilde{P}} \subseteq \Omega_{-10\ell}\},$$

then we can decompose the collection $\mathbb{P}_{\widetilde{P}'}''$ further, as follows:

$$(4.25) \quad \mathbb{P}_{\widetilde{P}'}'' = \bigcup_{d'' \in \mathbb{N}} \mathbb{P}_{\widetilde{P}', d''}'' ,$$

where

$$(4.26) \quad \mathbb{P}_{\widetilde{P}', d''}'' := \{P'' \in \mathbb{P}_{\widetilde{P}'}'' : 2^\ell I_{\widetilde{P}'} \times 2^{d''} I_{P''} \subseteq \widetilde{\Omega}_{-10\ell}\}$$

and d'' is maximal with this property.

Now we apply both the decompositions of $\widetilde{\mathbb{P}}'$ and $\mathbb{P}''_{\widetilde{P}'}$ defined in (4.10), (4.25) at the same time, and split the trilinear form $\Lambda_{\widetilde{\mathbb{P}}, II}^{\varepsilon, \ell}(f_1, f_2, f_3)$ into

$$(4.27) \quad \Lambda_{\widetilde{\mathbb{P}}, II}^{\varepsilon, \ell}(f_1, f_2, f_3) = \sum_{k' \in \mathbb{Z}} \sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}'_{k'}} |C_{Q_{\widetilde{P}}}^{\varepsilon}| |I_{\widetilde{P}'}| \sum_{d'' \in \mathbb{N}} \sum_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}} \frac{1}{|I_{P''}|^{\frac{1}{2}}} \\ \times \left| \left\langle \frac{f_1, \Phi_{\widetilde{P}'_1}^1}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_1}^1 \right\rangle \left\langle \frac{f_2, \Phi_{\widetilde{P}'_2}^2}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_2}^2 \right\rangle \left\langle \frac{f_3, \Phi_{\widetilde{P}'_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_3}^{3, \ell} \right\rangle \right|.$$

In the inner sum of the above (4.27), since $2^\ell I_{\widetilde{P}'} \times 2^{d''} I_{P''} \subseteq \widetilde{\Omega}_{-10\ell}$, $\text{supp}(\Phi_{\widetilde{P}'_3}^{3, \ell}) \subseteq 2^\ell I_{\widetilde{P}'}$ and $\text{supp} f_3 \subseteq E'_3 \subseteq \mathbb{R}^2 \setminus U$, we can assume hereafter in this subsection that

$$(4.28) \quad |f_3| \leq \chi_{E'_3} \chi_{2^\ell I_{\widetilde{P}'}} \chi_{(2^{d''} I_{P''})^c}.$$

By using Proposition 3.5 and (4.13), we derive from (4.27) the following estimates

$$(4.29) \quad \Lambda_{\widetilde{\mathbb{P}}, II}^{\varepsilon, \ell}(f_1, f_2, f_3) \\ \lesssim \sum_{k' \in \mathbb{Z}} C_{k'}^{\varepsilon} \sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}'_{k'}} |I_{\widetilde{P}'}| \sum_{d'' \in \mathbb{N}} \left[\prod_{j=1}^2 (\text{energy}_j((\langle \frac{f_j, \Phi_{\widetilde{P}'_j}^j}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_j}^j)_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}))^{1-\theta_j} \right. \\ \times (\text{size}_j((\langle \frac{f_j, \Phi_{\widetilde{P}'_j}^j}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_j}^j)_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}))^{\theta_j}] (\text{size}_3((\langle \frac{f_3, \Phi_{\widetilde{P}'_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_3}^{3, \ell})_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}))^{\theta_3} \\ \times (\text{energy}_3((\langle \frac{f_3, \Phi_{\widetilde{P}'_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_3}^{3, \ell})_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}))^{1-\theta_3} \\ \left. \right]$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the inner sum in the right-hand side of (4.29), note that $I_{\widetilde{P}} \subseteq \Omega_{-10\ell}$, $P'' \in \mathbb{P}''_{\widetilde{P}', d''}$ and f_3 satisfies (4.28), we apply the *size* estimates in Lemma 3.6 and get for each $\widetilde{P}' \in \widetilde{\mathbb{P}}'_{k'}$ and $d'' \in \mathbb{N}$,

$$(4.30) \quad \text{size}_1((\langle \frac{f_1, \Phi_{\widetilde{P}'_1}^1}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_1}^1)_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}) \lesssim \sup_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{f_1, \Phi_{\widetilde{P}'_1}^1}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M| dx \lesssim 2^{11\ell+d''} |E_1|,$$

$$(4.31) \quad \text{size}_2((\langle \frac{f_2, \Phi_{\widetilde{P}'_2}^2}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_2}^2)_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}) \lesssim \sup_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{f_2, \Phi_{\widetilde{P}'_2}^2}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M| dx \lesssim 2^{11\ell+d''} |E_2|,$$

$$(4.32) \quad \text{size}_3((\langle \frac{f_3, \Phi_{\widetilde{P}'_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P''_3}^{3, \ell})_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}}) \lesssim \sup_{P'' \in \mathbb{P}''_{\widetilde{P}', d''}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{f_3, \Phi_{\widetilde{P}'_3}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}} \right| |\widetilde{\chi}_{I_{P''}}^M| dx \lesssim 2^{-(M-100)d''},$$

where $M > 0$ is arbitrarily large. Similar to the energy estimates obtained in (4.18), (4.19) and (4.20), by applying the *energy* estimates in Lemma 3.7 and Hölder estimates, we have for each $\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}$ and $d'' \in \mathbb{N}$,

$$(4.33) \quad \text{energy}_1((\langle \frac{f_1, \Phi_{\widetilde{P}'}}^1}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_1''}^1))_{P'' \in \mathbb{P}_{\widetilde{P}', d''}''} \lesssim (\int_{E_1} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2)^{\frac{1}{2}},$$

$$(4.34) \quad \text{energy}_2((\langle \frac{f_2, \Phi_{\widetilde{P}'}}^2}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_2''}^2))_{P'' \in \mathbb{P}_{\widetilde{P}', d''}''} \lesssim (\int_{E_2} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2)^{\frac{1}{2}},$$

$$(4.35) \quad \text{energy}_3((\langle \frac{f_3, \Phi_{\widetilde{P}'}}^{3, \ell}}{|I_{\widetilde{P}'}|^{\frac{1}{2}}}, \Phi_{P_3''}^3))_{P'' \in \mathbb{P}_{\widetilde{P}', d''}''} \lesssim (\int_{E_3} \frac{\widetilde{\chi}_{I_{\widetilde{P}'}}^{100, \ell}(x_1)}{|I_{\widetilde{P}'}|} dx_1 dx_2)^{\frac{1}{2}},$$

where the approximate cutoff function $\widetilde{\chi}_{I_{\widetilde{P}'}}^{100, \ell}(x_1)$ decays rapidly (of order 100) away from the interval $I_{\widetilde{P}'}$ at scale $|I_{\widetilde{P}'}|$ and satisfies additional property that $\text{supp } \widetilde{\chi}_{I_{\widetilde{P}'}}^{100, \ell} \subseteq 2^\ell I_{\widetilde{P}'}$.

Now we insert the size and energy estimates (4.30)-(4.35) into (4.29), by using the estimates (4.13), (4.22) and Hölder inequality, we get

$$(4.36) \quad \begin{aligned} \Lambda_{\widetilde{\mathbb{P}}, II}^{\varepsilon, \ell}(f_1, f_2, f_3) &\lesssim 2^{11\ell} |E_1|^{\theta_1} |E_2|^{\theta_2} \sum_{k' \in \mathbb{Z}} C_{k'}^\varepsilon \sum_{d'' \in \mathbb{N}} 2^{-(M\theta_3 - 100)d''} \\ &\times \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_1} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_1}{2}} \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_2} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100} dx \right)^{\frac{1-\theta_2}{2}} \left(\sum_{\widetilde{P}' \in \widetilde{\mathbb{P}}_{k'}} \int_{E_3} \widetilde{\chi}_{I_{\widetilde{P}'}}^{100, \ell} dx \right)^{\frac{1-\theta_3}{2}} \\ &\lesssim_{\varepsilon, \theta_1, \theta_2, \theta_3, M} 2^{11\ell} |E_1|^{\frac{1+\theta_1}{2}} |E_2|^{\frac{1+\theta_2}{2}} \sum_{d'' \in \mathbb{N}} 2^{-(M\theta_3 - 100)d''}. \end{aligned}$$

for every $\ell \in \mathbb{N}$ and $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponents $\frac{2}{1+\theta_1} = p_1$ strictly larger than 1 and close to 1 and $\frac{2}{1+\theta_2} = p_2$ strictly smaller than 2 and close to 2. The series over $d'' \in \mathbb{N}$ in (4.36) is summable if we choose M large enough (say, $M \simeq 200\theta_3^{-1}$). We finally get the estimate

$$(4.37) \quad \Lambda_{\widetilde{\mathbb{P}}, II}^{\varepsilon, \ell}(f_1, f_2, f_3) \lesssim_{\varepsilon, p, p_1, p_2} 2^{11\ell} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for every $\ell \in \mathbb{N}$, $\varepsilon > 0$ and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

4.3. Conclusions. By inserting the estimates (4.9), (4.24) and (4.37) into (4.3), we finally get

$$(4.38) \quad |\Lambda_{\widetilde{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3)| \lesssim_{\varepsilon, p, p_1, p_2} \sum_{\ell \in \mathbb{N}} 2^{-100\ell} 2^{12\ell} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}} \lesssim_{\varepsilon, p, p_1, p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for any $\varepsilon > 0$, which completes the proof of Proposition 2.17 for the model operators $\Pi_{\widetilde{\mathbb{P}}}^\varepsilon$.

This concludes the proof of Theorem 1.3.

5. PROOF OF THEOREM 1.4

In this section, we prove Theorem 1.4 by carrying out the proof of Proposition 2.17 for model operators $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$ defined in (2.54) with $\vec{\mathbb{P}} = \mathbb{P}' \times \mathbb{P}''$.

Fix indices p_1, p_2, p as in the hypothesis of Proposition 2.17. Fix arbitrary measurable sets E_1, E_2, E_3 of finite measure (by using the scaling invariance of $\tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon$, we can assume further that $|E_3| = 1$). Our goal is to find $E'_3 \subseteq E_3$ with $|E'_3| \simeq |E_3| = 1$ such that, for any functions $|f_1| \leq \chi_{E_1}$, $|f_2| \leq \chi_{E_2}$ and $|f_3| \leq \chi_{E'_3}$, one has the corresponding trilinear forms $\tilde{\Lambda}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3)$ defined by

$$(5.1) \quad \tilde{\Lambda}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3) := \int_{\mathbb{R}^2} \tilde{\Pi}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2)(x) f_3(x) dx$$

satisfy estimates

$$(5.2) \quad |\tilde{\Lambda}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3)| = \left| \sum_{\vec{P} \in \vec{\mathbb{P}}} \frac{\tilde{C}_{Q_{\vec{P}}}^\varepsilon}{|I_{\vec{P}}|^{\frac{1}{2}}} \langle f_1, \Phi_{\vec{P}_1}^1 \rangle \langle f_2, \Phi_{\vec{P}_2}^2 \rangle \langle f_3, \Phi_{\vec{P}_3}^3 \rangle \right| \lesssim_{\varepsilon, p, p_1, p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}},$$

where p_1 is larger than but close to 1, while p_2 is smaller than but close to 2.

We define the exceptional set

$$(5.3) \quad \Omega := \{x \in \mathbb{R}^2 : MM(\frac{\chi_{E_1}}{|E_1|})(x) > C\} \cup \{x \in \mathbb{R}^2 : MM(\frac{\chi_{E_2}}{|E_2|})(x) > C\}.$$

It is clear that $|\Omega| < \frac{1}{10}$ if C is a large enough constant, which we fix from now on. Then, we define $E'_3 := E_3 \setminus \Omega$ and observe that $|E'_3| \simeq 1$.

Now we estimate the trilinear form $\tilde{\Lambda}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3)$ defined in (5.1) by two terms as follows:

$$(5.4) \quad \begin{aligned} |\tilde{\Lambda}_{\vec{\mathbb{P}}}^\varepsilon(f_1, f_2, f_3)| &\lesssim \sum_{\substack{\vec{P} \in \vec{\mathbb{P}}: \\ I_{\vec{P}} \cap \Omega^c \neq \emptyset}} \frac{|\tilde{C}_{Q_{\vec{P}}}^\varepsilon|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi_{\vec{P}_1}^1 \rangle| |\langle f_2, \Phi_{\vec{P}_2}^2 \rangle| |\langle f_3, \Phi_{\vec{P}_3}^3 \rangle| \\ &\quad + \sum_{\substack{\vec{P} \in \vec{\mathbb{P}}: \\ I_{\vec{P}} \cap \Omega^c = \emptyset}} \frac{|\tilde{C}_{Q_{\vec{P}}}^\varepsilon|}{|I_{\vec{P}}|^{\frac{1}{2}}} |\langle f_1, \Phi_{\vec{P}_1}^1 \rangle| |\langle f_2, \Phi_{\vec{P}_2}^2 \rangle| |\langle f_3, \Phi_{\vec{P}_3}^3 \rangle| \\ &=: \tilde{\Lambda}_{\vec{\mathbb{P}}, I}^\varepsilon(f_1, f_2, f_3) + \tilde{\Lambda}_{\vec{\mathbb{P}}, II}^\varepsilon(f_1, f_2, f_3). \end{aligned}$$

5.1. Estimates for trilinear form $\tilde{\Lambda}_{\vec{\mathbb{P}}, I}^\varepsilon(f_1, f_2, f_3)$. We can decompose the collection $\vec{\mathbb{P}}'$ of tri-tiles into

$$(5.5) \quad \mathbb{P}' = \bigcup_{k' \in \mathbb{Z}} \mathbb{P}'_{k'},$$

where

$$(5.6) \quad \mathbb{P}'_{k'} := \{P' \in \mathbb{P}' : \ell(Q_{P'}) = 2^{k'}\}.$$

As a consequence, we can split the trilinear form $\tilde{\Lambda}_{\mathbb{P},I}^\varepsilon(f_1, f_2, f_3)$ into

$$(5.7) \quad \tilde{\Lambda}_{\mathbb{P},I}^\varepsilon(f_1, f_2, f_3) = \sum_{k' \in \mathbb{Z}} \sum_{\substack{\vec{P} \in \mathbb{P}'_{k'} \times \mathbb{P}''; \\ I_{\vec{P}} \cap \Omega^c \neq \emptyset}} |\tilde{C}_{Q_{\vec{P}}}^\varepsilon| \frac{|I_{P'}|}{|I_{P''}|^{\frac{1}{2}}} \left| \left\langle \frac{\langle f_1, \Phi_{P'_1}^1 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_1}^1 \right\rangle \right| \\ \times \left| \left\langle \frac{\langle f_2, \Phi_{P'_2}^2 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_2}^2 \right\rangle \right| \left| \left\langle \frac{\langle f_3, \Phi_{P'_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_3}^3 \right\rangle \right|.$$

Note that by Lemma 2.10, we can estimate the Fourier coefficients $\tilde{C}_{Q_{\vec{P}}}^\varepsilon := \tilde{C}_{Q_{\vec{P}}, \vec{0}, \vec{0}, \vec{0}}^\varepsilon$ for each $\vec{P} \in \mathbb{P}'_{k'} \times \mathbb{P}''$ ($k' \in \mathbb{Z}$) by

$$(5.8) \quad |\tilde{C}_{Q_{\vec{P}}}^\varepsilon| \lesssim \tilde{C}_{k'}^\varepsilon := \langle k' \rangle^{-(1+\varepsilon)} = (1 + |k'|^2)^{-\frac{1+\varepsilon}{2}}.$$

For each fixed $P' \in \mathbb{P}'$, we define the sub-collection $\mathbb{P}''_{P'}$ of \mathbb{P}'' by

$$\mathbb{P}''_{P'} := \{P'' \in \mathbb{P}'' : I_{\vec{P}} \cap \Omega^c \neq \emptyset\}.$$

Therefore, by using Proposition 3.5, we derive the following estimates

$$(5.9) \quad \tilde{\Lambda}_{\mathbb{P},I}^\varepsilon(f_1, f_2, f_3) \lesssim \sum_{k' \in \mathbb{Z}} \tilde{C}_{k'}^\varepsilon \sum_{P' \in \mathbb{P}'_{k'}} |I_{P'}| \left[\prod_{j=1}^2 (\text{energy}_j((\langle \frac{\langle f_j, \Phi_{P'_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_j}^j)_{P'' \in \mathbb{P}''_{P'}}))^{1-\theta_j} \right. \\ \times (\text{size}_j((\langle \frac{\langle f_j, \Phi_{P'_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_j}^j)_{P'' \in \mathbb{P}''_{P'}}))^{\theta_j}] (\text{size}_3((\langle \frac{\langle f_3, \Phi_{P'_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_3}^3)_{P'' \in \mathbb{P}''_{P'}}))^{\theta_3} \\ \times (\text{energy}_3((\langle \frac{\langle f_3, \Phi_{P'_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_3}^3)_{P'' \in \mathbb{P}''_{P'}}))^{1-\theta_3}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the right-hand side of (5.9), note that $I_{\vec{P}} \cap \Omega^c \neq \emptyset$ and $\text{supp } f_3 \subseteq E'_3$, we apply the *size* estimates in Lemma 3.6 and get for each $P' \in \mathbb{P}'_{k'}$ and $j = 1, 2, 3$,

$$(5.10) \quad \text{size}_j((\langle \frac{\langle f_j, \Phi_{P'_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_j}^j)_{P'' \in \mathbb{P}''_{P'}}) \lesssim \sup_{P'' \in \mathbb{P}''_{P'}} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} \left| \frac{\langle f_j, \Phi_{P'_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}} \right| |\tilde{\chi}_{I_{P''}}^M| dx \lesssim |E_j|,$$

where $M > 0$ is sufficiently large. By applying the *energy* estimates in Lemma 3.7, we have for each $P' \in \mathbb{P}'_{k'}$ and $j = 1, 2, 3$,

$$(5.11) \quad \text{energy}_j((\langle \frac{\langle f_j, \Phi_{P'_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P'_j}^j)_{P'' \in \mathbb{P}''_{P'}}) \lesssim \frac{1}{|I_{P'}|^{\frac{1}{2}}} \left(\int_{\mathbb{R}} |\langle f_j, \Phi_{P'_j}^j \rangle|^2 dx_2 \right)^{\frac{1}{2}}.$$

Now we insert the size and energy estimates (5.10), (5.11) into (5.9) and get

$$(5.12) \quad \tilde{\Lambda}_{\mathbb{P},I}^\varepsilon(f_1, f_2, f_3) \lesssim |E_1|^{\theta_1} |E_2|^{\theta_2} \sum_{k' \in \mathbb{Z}} \tilde{C}_{k'}^\varepsilon \sum_{P' \in \mathbb{P}'_{k'}} \left\{ \prod_{j=1}^3 \left(\int_{\mathbb{R}} |\langle f_j, \Phi_{P'_j}^j \rangle|^2 dx_2 \right)^{\frac{1-\theta_j}{2}} \right\}.$$

Observe that for any different tri-tiles $\bar{P}' \in \mathbb{P}'_{k'}$ and $\bar{\bar{P}}' \in \mathbb{P}'_{k'}$, one has $I_{\bar{P}'} \cap I_{\bar{\bar{P}}'} = \emptyset$, or otherwise, one has $I_{\bar{P}'} = I_{\bar{\bar{P}}'}$, but $\omega_{\bar{P}'_j} \cap \omega_{\bar{\bar{P}}'_j} = \emptyset$ for every $j = 1, 2, 3$. By taking advantage of such orthogonality in L^2 of the wave packets $\Phi_{P'_j}^j$ corresponding to the tiles P'_j ($j = 1, 2, 3$), one has that for any function $F \in L^2(\mathbb{R})$ and $k' \in \mathbb{Z}$,

$$\begin{aligned}
(5.13) \quad \left\| \sum_{P' \in \mathbb{P}'_{k'}} \langle F, \Phi_{P'_j}^j \rangle \Phi_{P'_j}^j \right\|_{L^2}^2 &\leq \sum_{\substack{\bar{P}', \bar{\bar{P}}' \in \mathbb{P}'_{k'}: \\ \omega_{\bar{P}'_j} = \omega_{\bar{\bar{P}}'_j}; I_{\bar{P}'} \cap I_{\bar{\bar{P}}'} = \emptyset}} |\langle F, \Phi_{\bar{P}'_j}^j \rangle| |\langle F, \Phi_{\bar{\bar{P}}'_j}^j \rangle| |\langle \Phi_{\bar{P}'_j}^j, \Phi_{\bar{\bar{P}}'_j}^j \rangle| \\
&\lesssim 2^{k'} \sum_{\bar{P}' \in \mathbb{P}'_{k'}} |\langle F, \Phi_{\bar{P}'_j}^j \rangle|^2 \sum_{\substack{\bar{\bar{P}}' \in \mathbb{P}'_{k'}: \\ \omega_{\bar{P}'_j} = \omega_{\bar{\bar{P}}'_j}; I_{\bar{P}'} \cap I_{\bar{\bar{P}}'} = \emptyset}} |\langle \tilde{\chi}_{I_{\bar{P}'}}^{1000}, \tilde{\chi}_{I_{\bar{\bar{P}}'}}^{1000} \rangle| \\
&\lesssim \sum_{\bar{P}' \in \mathbb{P}'_{k'}} |\langle F, \Phi_{\bar{P}'_j}^j \rangle|^2 \sum_{\substack{\bar{\bar{P}}' \in \mathbb{P}'_{k'}: \\ \omega_{\bar{P}'_j} = \omega_{\bar{\bar{P}}'_j}; I_{\bar{P}'} \cap I_{\bar{\bar{P}}'} = \emptyset}} \left(1 + \frac{\text{dist}(I_{\bar{P}'}, I_{\bar{\bar{P}}'})}{|I_{\bar{P}'}|}\right)^{-100} \\
&\lesssim \sum_{P' \in \mathbb{P}'_{k'}} |\langle F, \Phi_{P'_j}^j \rangle|^2,
\end{aligned}$$

from which we deduce the following Bessel-type inequality

$$\begin{aligned}
(5.14) \quad \sum_{P' \in \mathbb{P}'_{k'}} |\langle F, \Phi_{P'_j}^j \rangle|^2 &= \left| \left\langle \sum_{P' \in \mathbb{P}'_{k'}} \langle F, \Phi_{P'_j}^j \rangle \Phi_{P'_j}^j, F \right\rangle \right| \\
&\leq \left\| \sum_{P' \in \mathbb{P}'_{k'}} \langle F, \Phi_{P'_j}^j \rangle \Phi_{P'_j}^j \right\|_{L^2} \cdot \|F\|_{L^2} \lesssim \|F\|_{L^2}^2,
\end{aligned}$$

where the implicit constants in the bounds are independent of $k' \in \mathbb{Z}$. Then, we can use Bessel-type inequality (5.14) and Hölder inequality to estimate the inner sum in the right-hand side of (5.12) by

$$\begin{aligned}
(5.15) \quad \sum_{P' \in \mathbb{P}'_{k'}} \left\{ \prod_{j=1}^3 \left(\int_{\mathbb{R}} |\langle f_j, \Phi_{P'_j}^j \rangle|^2 dx_2 \right)^{\frac{1-\theta_j}{2}} \right\} &\lesssim \prod_{j=1}^3 \left(\int_{\mathbb{R}} \sum_{P' \in \mathbb{P}'_{k'}} |\langle f_j, \Phi_{P'_j}^j \rangle|^2 dx_2 \right)^{\frac{1-\theta_j}{2}} \\
&\lesssim \prod_{j=1}^3 \|f_j\|_{L^2(\mathbb{R}^2)}^{1-\theta_j} \lesssim |E_1|^{\frac{1-\theta_1}{2}} |E_2|^{\frac{1-\theta_2}{2}}.
\end{aligned}$$

Combining the estimates (5.8), (5.12) and (5.15), we arrive at

$$(5.16) \quad \tilde{\Lambda}_{\mathbb{P}, I}^\varepsilon(f_1, f_2, f_3) \lesssim |E_1|^{\theta_1} |E_2|^{\theta_2} |E_1|^{\frac{1-\theta_1}{2}} |E_2|^{\frac{1-\theta_2}{2}} \sum_{k' \in \mathbb{Z}} \tilde{C}_{k'}^\varepsilon \lesssim_{\varepsilon, \theta_1, \theta_2, \theta_3} |E_1|^{\frac{1+\theta_1}{2}} |E_2|^{\frac{1+\theta_2}{2}}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponents $\frac{2}{1+\theta_1} = p_1$ strictly larger than 1 and close to 1 and $\frac{2}{1+\theta_2} = p_2$ strictly smaller

than 2 and close to 2. We finally get the estimate

$$(5.17) \quad \tilde{\Lambda}_{\mathbb{P},I}^\varepsilon(f_1, f_2, f_3) \lesssim_{\varepsilon,p,p_1,p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for every $\varepsilon > 0$ and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

5.2. Estimates for trilinear form $\tilde{\Lambda}_{\mathbb{P},II}^\varepsilon(f_1, f_2, f_3)$. For each fixed $P' \in \mathbb{P}'$, we define the corresponding sub-collection of \mathbb{P}'' by

$$\mathbb{P}_{P'}'' := \{P'' \in \mathbb{P}'' : I_{\bar{P}} \subseteq \Omega\},$$

then we can decompose the collection $\mathbb{P}_{P'}''$ further, as follows:

$$(5.18) \quad \mathbb{P}_{P'}'' = \bigcup_{\mu \in \mathbb{N}} \mathbb{P}_{P',\mu}'',$$

where

$$(5.19) \quad \mathbb{P}_{P',\mu}'' := \{P'' \in \mathbb{P}_{P'}'' : Dil_{2^\mu}(I_{P'} \times I_{P''}) \subseteq \Omega\}$$

and μ is maximal with this property. By $Dil_{2^\mu}(I_{\bar{P}})$ we denote the rectangle having the same center as the original $I_{\bar{P}}$ but whose side-lengths are 2^μ times larger.

Now we apply both the decompositions of \mathbb{P}' and $\mathbb{P}_{P'}''$ defined in (5.5), (5.18) at the same time, and split the trilinear form $\tilde{\Lambda}_{\mathbb{P},II}^\varepsilon(f_1, f_2, f_3)$ into

$$(5.20) \quad \begin{aligned} \tilde{\Lambda}_{\mathbb{P},II}^\varepsilon(f_1, f_2, f_3) &= \sum_{k' \in \mathbb{Z}} \sum_{P' \in \mathbb{P}_{k'}'} |\tilde{C}_{Q_{\bar{P}}}^\varepsilon| |I_{P'}| \sum_{\mu \in \mathbb{N}} \sum_{P'' \in \mathbb{P}_{P',\mu}''} \frac{1}{|I_{P''}|^{\frac{1}{2}}} \\ &\quad \times \left| \left\langle \frac{\langle f_1, \Phi_{P_1}^1 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_1}^1 \right\rangle \left\langle \frac{\langle f_2, \Phi_{P_2}^2 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_2}^2 \right\rangle \left\langle \frac{\langle f_3, \Phi_{P_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_3}^3 \right\rangle \right|. \end{aligned}$$

In the inner sum of the above (5.20), since $Dil_{2^\mu}(I_{P'} \times I_{P''}) \subseteq \Omega$, and $supp f_3 \subseteq E'_3 \subseteq \mathbb{R}^2 \setminus \Omega$, we get that

$$(5.21) \quad |f_3| \leq \chi_{E'_3} \chi_{(Dil_{2^\mu}(I_{P'} \times I_{P''}))^c} = \chi_{E'_3} \{ \chi_{(2^\mu I_{P'})^c} + \chi_{(2^\mu I_{P''})^c} - \chi_{(2^\mu I_{P'})^c} \chi_{(2^\mu I_{P''})^c} \},$$

and hence we can assume hereafter in this subsection that

$$(5.22) \quad |f_3| \leq \chi_{E'_3} \chi_{(2^\mu I_{P'})^c},$$

and the other two terms can be handled similarly.

By using Proposition 3.5 and (5.8), we derive from (5.20) the following estimates

$$(5.23) \quad \begin{aligned} &\tilde{\Lambda}_{\mathbb{P},II}^\varepsilon(f_1, f_2, f_3) \\ &\lesssim \sum_{k' \in \mathbb{Z}} \tilde{C}_{k'}^\varepsilon \sum_{P' \in \mathbb{P}_{k'}'} |I_{P'}| \sum_{\mu \in \mathbb{N}} \left[\prod_{j=1}^2 (energy_j(\left\langle \frac{\langle f_j, \Phi_{P_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_j}^j \right\rangle)_{P'' \in \mathbb{P}_{P',\mu}''})^{1-\theta_j} \right. \\ &\quad \times (size_j(\left\langle \frac{\langle f_j, \Phi_{P_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_j}^j \right\rangle)_{P'' \in \mathbb{P}_{P',\mu}''})^{\theta_j} \left. (size_3(\left\langle \frac{\langle f_3, \Phi_{P_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_3}^3 \right\rangle)_{P'' \in \mathbb{P}_{P',\mu}''})^{\theta_3} \right. \\ &\quad \times (energy_3(\left\langle \frac{\langle f_3, \Phi_{P_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_3}^3 \right\rangle)_{P'' \in \mathbb{P}_{P',\mu}''})^{1-\theta_3} \end{aligned}$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

To estimate the inner sum in the right-hand side of (5.23), note that $I_{\tilde{P}} \subseteq \Omega$, $P'' \in \mathbb{P}_{P', \mu}''$ and f_3 satisfies (5.22), we apply the *size* estimates in Lemma 3.6 and get for each $P' \in \mathbb{P}_{k'}'$ and $\mu \in \mathbb{N}$,

$$(5.24) \quad size_1((\langle \frac{\langle f_1, \Phi_{P_1}^1 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_1}^1 \rangle)_{P'' \in \mathbb{P}_{P', \mu}''}) \lesssim \sup_{P'' \in \mathbb{P}_{P', \mu}''} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} |\frac{\langle f_1, \Phi_{P_1}^1 \rangle}{|I_{P'}|^{\frac{1}{2}}}| \tilde{\chi}_{I_{P''}}^M dx \lesssim 2^{2\mu} |E_1|,$$

$$(5.25) \quad size_2((\langle \frac{\langle f_2, \Phi_{P_2}^2 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_2}^2 \rangle)_{P'' \in \mathbb{P}_{P', \mu}''}) \lesssim \sup_{P'' \in \mathbb{P}_{P', \mu}''} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} |\frac{\langle f_2, \Phi_{P_2}^2 \rangle}{|I_{P'}|^{\frac{1}{2}}}| \tilde{\chi}_{I_{P''}}^M dx \lesssim 2^{2\mu} |E_2|,$$

$$(5.26) \quad size_3((\langle \frac{\langle f_3, \Phi_{P_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_3}^3 \rangle)_{P'' \in \mathbb{P}_{P', \mu}''}) \lesssim \sup_{P'' \in \mathbb{P}_{P', \mu}''} \frac{1}{|I_{P''}|} \int_{\mathbb{R}} |\frac{\langle f_3, \Phi_{P_3}^3 \rangle}{|I_{P'}|^{\frac{1}{2}}}| \tilde{\chi}_{I_{P''}}^M dx \lesssim 2^{-N\mu},$$

where $M > 0$ and $N > 0$ are arbitrarily large. By applying the *energy* estimates in Lemma 3.7, we have for each $P' \in \mathbb{P}_{k'}'$, $\mu \in \mathbb{N}$ and $j = 1, 2, 3$,

$$(5.27) \quad energy_j((\langle \frac{\langle f_j, \Phi_{P_j}^j \rangle}{|I_{P'}|^{\frac{1}{2}}}, \Phi_{P_j}^j \rangle)_{P'' \in \mathbb{P}_{P', \mu}''}) \lesssim \frac{1}{|I_{P'}|^{\frac{1}{2}}} (\int_{\mathbb{R}} |\langle f_j, \Phi_{P_j}^j \rangle|^2 dx_2)^{\frac{1}{2}}.$$

Now we insert the size and energy estimates (5.24)-(5.27) into (5.23), by using the estimates (5.8) and (5.15), we derive that

$$(5.28) \quad \begin{aligned} \tilde{\Lambda}_{\mathbb{P}, II}^\varepsilon(f_1, f_2, f_3) &\lesssim |E_1|^{\theta_1} |E_2|^{\theta_2} \sum_{k' \in \mathbb{Z}} \tilde{C}_{k'}^\varepsilon \sum_{\mu \in \mathbb{N}} 2^{-(N\theta_3-2)\mu} \sum_{P' \in \mathbb{P}_{k'}'} \{ \prod_{j=1}^3 (\int_{\mathbb{R}} |\langle f_j, \Phi_{P_j}^j \rangle|^2 dx_2)^{\frac{1-\theta_j}{2}} \} \\ &\lesssim_{\varepsilon, \theta_1, \theta_2, \theta_3, N} |E_1|^{\frac{1+\theta_1}{2}} |E_2|^{\frac{1+\theta_2}{2}} \sum_{\mu \in \mathbb{N}} 2^{-(N\theta_3-2)\mu}. \end{aligned}$$

for every $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

By taking θ_1 sufficiently close to 1 and θ_2 sufficiently close to 0, one can make the exponents $\frac{2}{1+\theta_1} = p_1$ strictly larger than 1 and close to 1 and $\frac{2}{1+\theta_2} = p_2$ strictly smaller than 2 and close to 2. The series over $\mu \in \mathbb{N}$ in (5.28) is summable if we choose N large enough (say, $N \simeq 4\theta_3^{-1}$). We finally get the estimate

$$(5.29) \quad \tilde{\Lambda}_{\mathbb{P}, II}^\varepsilon(f_1, f_2, f_3) \lesssim_{\varepsilon, p, p_1, p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for any $\varepsilon > 0$ and p, p_1, p_2 satisfy the hypothesis of Proposition 2.17.

5.3. Conclusions. By inserting the estimates (5.17) and (5.29) into (5.4), we finally get

$$(5.30) \quad |\tilde{\Lambda}_{\mathbb{P}}^\varepsilon(f_1, f_2, f_3)| \lesssim_{\varepsilon, p, p_1, p_2} |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}}$$

for any $\varepsilon > 0$, which completes the proof of Proposition 2.17 for the model operators $\tilde{\Pi}_{\mathbb{P}}^\varepsilon$.

This concludes the proof of Theorem 1.4.

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